Basic concepts in the geometry of Banach spaces

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Contents

1 Introduction

In this introductory chapter we present results and concepts which are often used in Banach space theory and will be used in articles in this Handbook without further reference. The material we treat, while familiar to experts in Banach space theory, has not made its way into introductory courses in functional analysis. The main purpose of this article is to make the subsequent articles accessible to anyone whose background includes basic graduate courses in analysis and functional analysis. Each section of this article is devoted to one aspect of Banach space theory.

Although this article can in no way be considered as an introductory course in Banach space theory, we do include indications of proof of some basic results in the hope that this will help the reader understand and get a feeling for the various concepts which are discussed. We reference only some of the (mostly introductory) books which treat the basic material we describe. Original sources are referenced in these books. We also mention some results which are not yet in elementary books on Banach spaces but which help to clarify the general picture. These generally were either discovered recently or are more difficult than the rest of the material. In these cases we refer to specific articles in this Handbook which treat the topic.

In general, we do not attach names to theorems (except when experts generally refer to the theorem with a name attached, such as the Hahn-Banach theorem, the Krein-Milman theorem, Rosenthal's ℓ_1 theorem,...) or give any historical background. Instead we refer to the books in the references as well as the articles in this Handbook.

2 Notations and special Banach spaces

Banach spaces will have either real or complex scalars. When the scalar field matters (for example, in results involving spectral theory or in theorems of an isometric nature or when analyticity plays a rôle), the scalar field is mentioned explicitly, but in the notation for special spaces the scalars are not specified.

Operators between Banach spaces are bounded and linear. An invertible operator T is called an *isomorphism.* Two norms on a vector space are called *equivalent* if the identity operator on X (with the two given norms) is an isomorphism. If $\|T\| = 1 = \|T^{-1}\|$, T is called an *isometric isomorphism* or simply an *isometry* and the domain and range of T are said to be *isometric.* We write $X \approx Y$ to denote that the spaces are isomorphic. To denote isometry we use the equal sign. An isomorphism from a Banach space onto itself is called an *automorphism.* A Banach space Y is said to be a *quotient* of the Banach space X if Y is isometric to X/Z for some closed subspace Z of X. By the open mapping theorem, Y is isomorphic to a quotient of X if there is an operator from X onto Y.

If $X \approx Y$, $d(X,Y)$ denotes the Banach-Mazur distance between the spaces, defined to be the infimum of $\|T\|\|T^{-1}\|$ as T ranges over all isomorphisms from X onto Y. So $d(X,Y) = 1$ if X and Y are isometric; the converse is true for finite dimensional spaces but not for infinite dimensional spaces. Note that the "triangle inequality" for the Banach-Mazur distance is submultiplicative rather than subadditive; that is, $d(X,Y) \le d(X,Z)d(Z,Y)$.

A *projection* is an idempotent operator. A subspace Z of X is said to be *complemented* if there is a projection from X onto Z. This is the case if and only if Z is closed and there is a closed subspace W of X so that $W \cap Z = \{0\}$ and $X = W + Z$; we then write $X = W \oplus Z$ and say that X is the *direct sum* of W and Z. In this case W is isomorphic to X/Z by the open mapping theorem.

We regard X as a subspace of X^{**} under the canonical embedding. Note that there is always a projection of norm one from X^{***} onto X^* (restrict the functionals on X^{**} to X).

An operator $T : X \to Y$ between Banach spaces is *compact;* respectively, *weakly compact;* if the image TB_X of the unit ball B_X of X has compact; respectively, weakly compact; closure in Y. If either X or Y is reflexive, then every operator from X to Y is weakly compact. The identity operator on X is compact if and only if X is locally compact if and only if X is finite dimensional.

A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called *weakly Cauchy* provided $\{x^*(x_n)\}_{n=1}^{\infty}$ converges for every x^* in X^*. Identifying a Banach space with a subspace of its bidual, and taking into account that the uniform boundedness principle implies that a weakly Cauchy sequence is bounded, we see that $\{x_n\}_{n=1}^{\infty}$ is weakly Cauchy in X if and only if it converges weak* in X^{**}. The space X is said to be *weakly sequentially complete* provided every weakly Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X converges weakly, which is the same as saying that the weak* limit of $\{x_n\}_{n=1}^{\infty}$ in X^{**} is in X itself.

Here is a list of special classical Banach spaces and other objects. The elementary theory of these can be found in beginning texts in real and functional analysis. A more detailed list of symbols, including some notation undefined in the text because we regard it as "standard", is contained in section 12.

$\mathbb{N}$ = The natural numbers.

$\mathbb{R}$ = The real numbers.

$\mathbb{C}$ = The complex numbers.

$\mathbb{T}$ = The unit circle in the complex plane.

$C(K;X)$ = Continuous functions f on the (usually) compact Hausdorff space K taking values in the (usually) Banach space X, normed by $\|f\| = \sup_{t\in K}\|f(t)\|$.

$C(K)$ = $C(K;X)$ when X is the scalar field.

$L_p(\mu)$ = The μ-measurable functions f for which $\|f\|_p = (\int |f|^p \,\mathrm{d}\mu)^{1/p} < \infty$. Here $0 < p < \infty$.

$L_\infty(\mu)$ = The μ-measurable essentially bounded functions, with norm $\|f\|_\infty = \inf_{\mu A=0} \sup |f_{|\tilde{A}}|$.

$L_p(0,1)$ = $L_p(\mu)$ when μ is Lebesgue measure on the unit interval.

$L_p(\mathbb{T})$ = $L_p(\mu)$ when μ is normalized Lebesgue measure on the unit circle.

$\ell_p(\Gamma)$ = $L_p(\mu)$ when μ is counting measure on the set Γ.

ℓ_p = $\ell_p(\Gamma)$ when $\Gamma = \mathbb{N}$.

ℓ_p^n = $\ell_p(\Gamma)$ when $\Gamma = \{1, 2, \ldots, n\}$.

c = The subspace of ℓ_∞ of scalar sequences which have a limit.

$c_0(\Gamma)$ = The closure in $\ell_\infty(\Gamma)$ of the scalar sequences which have finite support.

c_0 = $c_0(\Gamma)$ when $\Gamma = \mathbb{N}$.

B_X = The closed unit ball of the Banach space X.

$B_X(x,r)$ = The closed ball of radius r with center x in the Banach space X; denoted also $B(x,r)$ when X is understood.

When $0 < p < 1$, the space $L_p(\mu)$ is not a Banach space except in the trivial cases that it is zero or one dimensional. The metric on $L_p(\mu)$, $0 < p < 1$, is given by $\rho(x,y) = \|x-y\|^p$.

If $\{X_n\}_{n=1}^\infty$ is a sequence of Banach spaces and $1 \le p \le \infty$, $\left(\sum X_n\right)_p$ is the Banach space of all sequences $\{x_n\}_{n=1}^\infty$ with $x_n \in X_n$ and $\|\{x_n\}_{n=1}^\infty\| := \|\{\|x_n\|\}_{n=1}^\infty\|_p < \infty$. $\left(\sum X_n\right)_0$ is the closed subspace of $\left(\sum X_n\right)_\infty$ for which $\|x_n\| \to 0$. When all X_n are the same space X we sometimes write $\ell_p(X)$ instead of $\left(\sum X\right)_p$. While this standard notation conflicts with the (also standard) notation $\ell_p(\Gamma)$ introduced above, no confusion should arise. Sometimes also uncountable direct sums will be considered (with the obvious meaning). The elementary duality relations $c_0^* = \ell_1$ and $\ell_p^* = \ell_q$, $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, yield that $(\sum X_n)_0^* = (\sum X_n^*)_1$ and $(\sum X_n)_p^* = (\sum X_n^*)_q$ for $1 \le p < \infty$. When the product $X \oplus Y$ (called the *direct sum* of X and Y) is given the norm $\|(x,y)\| = \|(\|x\|, \|y\|)\|_p$, we write the resulting space $X \oplus_p Y$. All these norms on $X \oplus Y$ are equivalent. When we are interested in $X \oplus Y$ only up to isomorphism we do not specify a particular norm.

Probability theory has had a profound impact on Banach space theory. Consequently, we shall use in places probabilistic terminology. For example, we call the function which is one on a set A and zero on the complement of A the *indicator function* and denote it by 1_A. We recall now some other standard concepts and results in probability theory which can be found in most introductory textbooks in the subject; in particular, in [10]. A *random variable* is a real valued measurable function on a probability space. A random variable g is said to be *Gaussian* or to have *Gaussian distribution* or *normal distribution* if its *distribution function* $\mathbb{P}[g \le t]$ is equal to $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{\frac{-(x-a)^2}{2\sigma^2}}\, \mathrm{d}x$. Here $\mathbb{P}$ denotes the underlying probability measure. The constant a is the *mean* or *expectation* or *expected value* of g, defined by $\mathbb{E}g = \int g \,\mathrm{d}\mathbb{P}$. The quantity σ^2 is the *variance* of g, defined to be $\mathbb{E}(g-a)^2$. If $a = 0$ and $\sigma = 1$, g is called a *standard Gaussian variable.*

A finite collection $\{f_n\}_{n=1}^N$ of random variables on the same probability space is called *independent* provided that $\mathbb{P} \cap_{n=1}^N [f_n \in A_n] = \prod_{n=1}^N \mathbb{P}[f_n \in A_n]$ for all Borel sets A_n. So $\mathbb{E}(f \cdot g) = \mathbb{E}f\mathbb{E}g$ if f and g are independent. An arbitrary collection of random variables is independent provided each finite subcollection of the collection is independent. Given a sequence $\{f_n\}_{n=1}^\infty$ of random variables with f_n defined on the probability space $(\Omega_n, \mathbb{P}_n)$, one can construct a sequence $\{g_n\}_{n=1}^\infty$ of independent random variables so that g_n has for each n the same distribution as f_n. Indeed, let $(\Omega, \mathbb{P})$ be the infinite product of the probability spaces $(\Omega_n, \mathbb{P}_n)$ and for $\omega = \{\omega_n\}_{n=1}^\infty$ in Ω set $g_n(\omega) = f_n(\omega_n)$. In particular, it is possible to define a sequence of independent random variables having standard Gaussian distribution on the infinite product $(0,1)^{\mathbb{N}}$ of $(0,1)$ (with the product of Lebesgue measure). By the isomorphism theorem for separable measure algebras ([18, p. 399]), $(0,1)^{\mathbb{N}}$ can be replaced by $(0,1)$ itself, but it is often more convenient to work on the product space.

The *characteristic function* of the random variable g is the function $\varphi : \mathbb{R} \to \mathbb{C}$ defined by $\varphi(t) = \mathbb{E}e^{itg}$. Useful algebraic identities include $\varphi(-t) = \overline{\varphi(t)}$; $\varphi_{ag+b}(t) = e^{ibt}\varphi(at)$; and, especially, $\varphi_{f+g} = \varphi_f \varphi_g$, valid when f and g are independent. The characteristic function of a standard Gaussian random variable is $e^{-\frac{t^2}{2}}$.

A key fact is that two random variables (possibly defined on different probability spaces) have the same distribution if and only if they have the same characteristic function. This is proved by an inversion formula [10, 2.3.a], another simple consequence of which is that the characteristic function of a random variable g is real valued if and only if g is *symmetric*; that is, g and $-g$ have the same distribution. A (by no means immediate) consequence of the inversion formula [10, 2.7] is that for $0 < r < 2$ the function $e^{-|t|^r}$ is the characteristic function of a (necessarily symmetric) random variable, called a symmetric r-*stable* random variable. The *tail distribution* $\mathbb{P}[|g| > t]$ of a symmetric r-stable random variable g is like t^{-r} as $t \to \infty$, so that $\|g\|_p < \infty$ for $p < r$, but $\|g\|_r = \infty$.

Another probabilistic notion that plays an important role in Banach space theory is that of *martingale.* First we recall the notion of *conditional expectation.* Let $\mathbb{P}$ be a probability measure on a σ-algebra $\mathcal{B}$. If $\mathcal{A}$ is a sub σ-algebra of $\mathcal{B}$ and f is a $\mathbb{P}$-integrable function, then by the Radon-Nikodým theorem there is a $\mathcal{A}$-measurable function g so that for each A in $\mathcal{A}$, $\int_A f \, d\mathbb{P} = \int_A g \, d\mathbb{P}$. The function g is called the *conditional expectation of f given $\mathcal{A}$* and is sometimes denoted by $\mathbb{E}(f|\mathcal{A})$. Suppose now that $\{f_n\}_{n=0}^{\infty}$ is a sequence of random variables on the same probability space and $\mathcal{B}_n$ is the smallest σ-algebra for which $f_0, f_1, \ldots, f_n$ are measurable. The sequence $\{f_n\}_{n=0}^{\infty}$ is called a *martingale* provided that for each n, $f_n = \mathbb{E}(f_{n+1}|\mathcal{B}_n)$. The sequence $\{d_n\}_{n=0}^{\infty}$ of differences; $d_0 = f_0$, $d_n = f_n - f_{n-1}$; is then called a *martingale difference sequence.* Notice that a sequence $\{d_n\}_{n=0}^{\infty}$ of independent random variables is a martingale difference sequence if and only if d_n has mean zero for all $n \geq 1$. A simple but important example of a martingale difference sequence which is not a sequence of independent random variables is the Haar system, discussed in section 4. One basic theorem about martingales, the *martingale convergence theorem* (see [10, 4.2.10]), states that **every L_1 bounded martingale converges a.e.**, which means that if $\{f_n\}_{n=0}^{\infty}$ is a martingale of $\mathbb{P}$-measurable functions and $\sup_n \mathbb{E}|f_n| < \infty$, then $\{f_n\}_{n=0}^{\infty}$ converges a.e. Moreover, if the martingale is uniformly integrable (see section 4, then it also converges in $L_1(\mathbb{P})$ (see [10, 4.5.3]).

3 Bases

Excepting [2] and [12], which are oriented to Banach space theory, few introductory texts in functional analysis treat Schauder bases. Nevertheless, bases are a very useful tool for investigating properties of Banach spaces.

We prove few statements in this section since most of the results are proved in [2] and [12]. The book [14] often contains only sketches of proofs. Chapter 2 of [21] contains enough details to be pleasant reading for the mature student or experienced mathematician who is not an expert in Banach space theory. All of these books contain exercises. Many in [21] are challenging even after peeking at the hints. Exercises in [14], as in this introductory article, are scattered throughout the text and are sometimes flagged by such expressions as “clearly”, “hence”, and the like.

A *Schauder basis* or simply a *basis* for a Banach space X is a sequence $\{x_n\}_{n=1}^\infty$ of vectors in X such that every vector in X has a unique representation of the form $\sum \alpha_n x_n$ with each α_n a scalar and where the sum is converges in the norm topology. The mapping $x \mapsto \alpha_n$ then defines for each n a linear functional x_n^* on X. One checks that the expression $!x! = \sup_n \|\sum_{k=1}^n x_k^*(x)x_k\|$ defines a stronger complete norm on X, so that $!\cdot!$ and $\|\cdot\|$ are equivalent by the open mapping theorem. One deduces from this that the biorthogonal functionals for a basis are necessarily continuous. Moreover, the biorthogonal functionals are a *basic sequence* in X^*; that is, they form a basis for their closed linear span. When it is useful to specify the biorthogonal functionals, we sometimes refer to the “basis” $\{x_n, x_n^*\}_{n=1}^\infty$.

A sequence $\{x_n\}_{n=1}^\infty$ is called *normalized* if each vector x_n has norm one and $\{x_n\}_{n=1}^\infty$ is called *seminormalized* if it is bounded and bounded away from zero. For most purposes it is sufficient to consider normalized or at least seminormalized basic sequences.

The *partial sum projections* of a basis $\{x_n, x_n^*\}_{n=1}^\infty$, defined for $n = 1, 2, \dots$ by $P_n x = \sum_{i=1}^n x_i^*(x)x_i$, are necessarily uniformly bounded and *converge strongly* to the identity; that is, $\|x - P_n x\| \to 0$ for each x in X. The supremum of the norms of these partial sum projections is called the *basis constant*. This quantitative notion is of interest also if we just consider finite basic sequences or bases for finite dimensional spaces. A sequence $\{x_n\}_{n=1}^\infty$ of nonzero vectors is basic with basis constant at most C if and only if for all $n < m$ (and all scalars α_i) the inequality $\|\sum_{i=1}^n \alpha_i x_i\| \le C\|\sum_{i=1}^m \alpha_i x_i\|$ holds. A *block basis* $\{y_j\}_{j=1}^\infty$ of the basis $\{x_n\}_{n=1}^\infty$ is a sequence of nonzero vectors of the form $y_j = \sum_{k=n_j+1}^{n_{j+1}} \alpha_k x_k$ for some sequence $n_1 < n_2 < \dots$. The basis constant of a block basis of $\{x_n\}_{n=1}^\infty$ is no larger than the basis constant of $\{x_n\}_{n=1}^\infty$.

A basis is called *monotone* provided that its basis constant is one. One can change to an equivalent norm, $|||\cdot|||$, in which the basis is monotone: just set $|||x||| = \sup_n \|P_n x\|$. In fact, if one defines instead $|||x||| = \sup_{n<m} \|P_m x - P_n x\|$ one gets an equivalent norm which is even *bimonotone*; that is, the basis is monotone and also all the complementary projections $I - P_n$ have norm one.

Often one constructs a basic sequence $\{x_n\}_{n=1}^\infty$ by induction and checks that it is basic by verifying for each n the inequality $\|\sum_{i=1}^n \alpha_i x_i\| \le C_n\|\sum_{i=1}^{n+1} \alpha_i x_i\|$ with $\sum_{n=1}^\infty (C_n - 1) < \infty$; the basis constant of $\{x_n\}_{n=1}^\infty$ is then at most $\prod_{n=1}^\infty C_n$. Here is one classical construction, which yields, for example, that **every infinite dimensional Banach space contains a basic sequence.** Given any sequence

C_n as above and having selected $x_1, \ldots, x_n$, take a finite set $S = S_n$ of norm one linear functionals on X which almost determine the norm on the linear span X_n of $x_1, \ldots, x_n$ in the sense that for each $x \in X_n$, $\|x\| \leq C_n \max_{x^* \in S} |x^*(x)|$. Now x_{n+1} can be any nonzero vector in the finite codimensional subspace $S_\perp$ of X. This construction allows considerable "play"–given $\epsilon > 0$, as long as you select x_{n+1} so that $\max_{x^* \in S} \|x_{n+1}\|^{-1} |x^*(x_{n+1})|$ is sufficiently small, the resulting sequence $\{x_n\}_{n=1}^\infty$ will have basis constant less than $\epsilon + \prod_{n=1}^\infty C_n$. This yields, for example, that **every weakly null, non-norm null sequence has a basic subsequence.**

This last result can be improved substantially. Two basic sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are *equivalent* provided that the map $Tx_n = y_n$ extends to an isomorphism from the closed span of $\{x_n\}_{n=1}^\infty$ onto the closed span of $\{y_n\}_{n=1}^\infty$; K-*equivalent* if $\|T\|\|T^{-1}\| \leq K$ (so, strangely, $\{x_n\}_{n=1}^\infty$ is 1-equivalent to $\{2x_n\}_{n=1}^\infty$, but usually basic sequences are normalized). The *principle of small perturbations* says that **if $\{x_n\}_{n=1}^\infty$ is a basic sequence in X and $\|x_n - y_n\| \to 0$ sufficiently quickly, then $\{y_n\}_{n=1}^\infty$ is a basic sequence which is equivalent to $\{x_n\}_{n=1}^\infty$.** To prove this, let $\{x_n^*\}_{n=1}^\infty$ be Hahn-Banach extensions of the functionals biorthogonal to $\{x_n\}_{n=1}^\infty$ to functionals in X^* and define an operator S on X by $Sx = \sum x_n^*(x)(y_n - x_n)$, so $\|S\| \leq \sum \|x_n^*\|\|y_n - x_n\|$. If $\|S\| < 1$, elementary considerations yield that $I + S$ is an automorphism of X which maps x_n to y_n. This also yields that if $\{x_n\}_{n=1}^\infty$ is a basis of X, then so is $\{y_n\}_{n=1}^\infty$. From this one sees that if $\{x_n\}_{n=1}^\infty$ is a basis for X with biorthogonal functionals $\{x_n^*\}_{n=1}^\infty$, if $\{y_n\}_{n=1}^\infty$ is a seminormalized sequence in X, and if for each k, $x_k^*(y_n) \to 0$ as $n \to \infty$, then $\{y_n\}_{n=1}^\infty$ has a subsequence which is equivalent to some block basis $\{z_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$. From this it follows that **every infinite dimensional subspace Y of a Banach space X with a basis $\{x_n\}_{n=1}^\infty$ contains, for every $K > 1$, a normalized basic sequence $\{z_n\}_{n=1}^\infty$ which is K-equivalent to a normalized block basis $\{y_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$.** Moreover, the small perturbation argument indicated above shows that the generated isomorphism which maps $\{y_n\}_{n=1}^\infty$ into Y extends to an automorphism on X.

To see how these basic facts about bases might be used, suppose that T is a noncompact operator from a subspace X_0 of a Banach space X with basis $\{x_n, x_n^*\}_{n=1}^\infty$ into a Banach space Y with basis $\{z_n\}_{n=1}^\infty$. Then there is a sequence $\{y_n\}_{n=1}^\infty$ in the unit ball of X_0 such that for some $\epsilon > 0$ and all $n \neq m$, $\|Ty_n - Ty_m\| > \epsilon$. By passing to a subsequence of differences of $\{y_n\}_{n=1}^\infty$ it can be assumed that for each k, $x_k^*(y_n) \to 0$ as $n \to \infty$. In view of what was discussed in the previous paragraph, by passing to another subsequence it can be assumed that $\{y_n\}_{n=1}^\infty$ is an arbitrarily small perturbation of a block basis of $\{x_n\}_{n=1}^\infty$. Repeating the same argument for $\{Ty_n\}_{n=1}^\infty$ in Y and normalizing at the end, we conclude that **if T is a noncompact operator from a subspace X_0 of a Banach space X with basis $\{x_n\}_{n=1}^\infty$ into a Banach space Y with basis $\{z_n\}_{n=1}^\infty$, then there are automorphisms U on X and V on**

Y and a normalized block basis $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ with $y_n \in U^{-1}X_0$ and such that $\{VTUy_n\}_{n=1}^{\infty}$ is a seminormalized block basis of $\{z_n\}_{n=1}^{\infty}$.

This last mentioned result gives considerable information in cases where all block bases of a basis can be characterized. The simplest examples of bases are provided by the unit vector basis $\{e_n\}_{n=1}^{\infty}$ for ℓ_p, $1 \le p < \infty$, and c_0. It is evident that in these spaces every normalized block basis of $\{e_n\}_{n=1}^{\infty}$ is 1-equivalent to $\{e_n\}_{n=1}^{\infty}$ and every seminormalized block basis of $\{e_n\}_{n=1}^{\infty}$ is equivalent to $\{e_n\}_{n=1}^{\infty}$. Thus we can conclude that **every operator from a subspace of ℓ_r into ℓ_p, $1 \le p < r < \infty$, is compact**. This shows in particular that **no infinite dimensional subspace of ℓ_r isomorphically embeds into ℓ_p when $1 \le p, r < \infty$ and $p \ne r$** (an isomorphism between infinite dimensional spaces cannot be compact since no infinite dimensional space is locally compact).

We should mention that, in contrast to the case of ℓ_p and ℓ_r, it generally is quite difficult to determine whether two spaces are isomorphic even when both are presented concretely as similar but different function spaces.

Another consequence of the principle of small perturbations is that ℓ_1 has the *Schur property,* which means that **every weakly convergent sequence in ℓ_1 is norm convergent.** Indeed, otherwise there would be a weakly null sequence $\{x_n\}_{n=1}^{\infty}$ in ℓ_1 which is bounded away from zero. The sequence $\{x_n\}_{n=1}^{\infty}$ is necessarily bounded, so the sequence $\{x_n\}_{n=1}^{\infty}$ would have a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which is equivalent to a block basis of the unit vector basis of ℓ_1 and thus equivalent to the unit vector basis of ℓ_1. This is a contradiction because the unit vector basis of ℓ_1 is not weakly null. An easy formal consequence of the fact that ℓ_1 has the Schur property is that every weakly Cauchy sequence in ℓ_1 is norm convergent. In particular, ℓ_1 is weakly sequentially complete.

The most natural basis for $L_p(0,1)$, $1 \le p < \infty$, is the *Haar system* $\{h_n\}_{n=0}^{\infty}$, where $h_0 = 1_{[0,1)}$, and for $n = 2^j + k$ with $j = 0, 1, \dots$ and $k = 0, 1, \dots, 2^j - 1$,

$$h_n = 1_{[k2^{-j},(2k+1)2^{-j-1})} - 1_{[(2k+1)2^{-j-1},(k+1)2^{-j})}.$$

It is easy to check that if $1 \le p \le \infty$, then for each n and all scalars α_i the inequality $\| \sum_{i=0}^{n} \alpha_i h_i \|_p \le \| \sum_{i=0}^{n+1} \alpha_i h_i \|_p$ holds, which means that the Haar system is a monotone basic sequence in $L_p(0,1)$. Since the linear span of the Haar system is dense in $L_p(0,1)$ when $p < \infty$, the Haar basis is a monotone basis for $L_p(0,1)$ when $1 \le p < \infty$.

The sequence $\{f_n\}_{n=0}^{\infty}$, where $f_0 = 1_{[0,1]}$ and for $n \ge 1$, f_n is the indefinite integral of the Haar function h_{n-1}, forms a monotone basis for $C[0,1]$. When normalized in the supremum norm, this is called the *Faber-Schauder basis.*

The trigonometric system $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ forms a basis for $L_p(\mathbb{T})$ when it is ordered $0, 1, -1, 2, -2, \ldots$ and $1 < p < \infty$. This follows from the boundedness of the Riesz projection in this range. The *Riesz projection* is defined formally by $\mathcal{R}(f) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta}$, where $\hat{f}(n) = (2\pi)^{-1}\int_0^{2\pi} f(t)e^{-int}\,\mathrm{d}t$ is the nth Fourier coefficient of the function f. The unboundedness of the Riesz projection on $L_1(\mathbb{T})$ and $C(\mathbb{T})$ yields that the trigonometric system is not a basis for these spaces.

A series $\sum x_n$ in a Banach space is said to *converge unconditionally* provided every rearrangement of the series converges. This is equivalent to $\sum x_{k_n}$ converges for each subsequence of $\{x_n\}_{n=1}^{\infty}$ and also to $\sum \pm x_n$ converges for each choice of signs $\pm$. Except in finite dimensional spaces, unconditional convergence is weaker than *absolute convergence;* that is, convergence of $\sum \|x_n\|$ (see section 8).

A basis $\{x_n\}_{n=1}^{\infty}$ is said to be an *unconditional basis* provided that $\sum \alpha_n x_n$ converges unconditionally whenever it converges. This is equivalent to saying that every permutation of $\{x_n\}_{n=1}^{\infty}$ is also a basis. If $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for the Banach space X and $\theta = \{\theta_n\}_{n=1}^{\infty}$ is a sequence of ±1's, define $S_\theta : X \to X$ by $S_\theta(\sum \alpha_n x_n) = \sum \theta_n \alpha_n x_n$. The supremum over all such θ of $\|S_\theta\|$ is finite, and is called the *unconditional constant* of the basis. One can define an equivalent norm $|||\cdot|||$ on X for which the unconditional basis has unconditional constant one by setting $|||x||| = \sup_\theta \|S_\theta x\|$. Such a norm is said to be *unconditionally monotone*. When the norm is unconditionally monotone, all permutations of the basis are monotone (sometimes this weaker condition is referred to as "unconditionally monotone").

A sequence which is an unconditional basis for its closed linear span is said to be *unconditionally basic*. Since a block basis of an unconditional basis is clearly unconditional (with unconditional constant no larger than that of the basis itself), the perturbation principle described above yields that **every infinite dimensional subspace of a space with unconditional basis contains an unconditionally basic sequence**.

The unit vector bases for ℓ_p, $1 \le p < \infty$, and c_0 are the simplest examples of unconditional bases. For $1 < p < \infty$, the Haar system forms an unconditional basis for $L_p(0,1)$. The "modern" proof of this proceeds via martingale theory. The exact unconditional constant of the Haar system in $L_p(0,1)$ is computed in [23]. In section 8 we point out that the trigonometric system is an unconditional basis in $L_p(\mathbb{T})$ only for $p = 2$.

A basis $\{x_n\}_{n=1}^{\infty}$ for X is *shrinking* provided the linear span of the biorthogonal functionals $\{x_n^*\}_{n=1}^{\infty}$ is (norm) dense in X^*, which is to say that $\{x_n^*\}_{n=1}^{\infty}$ is a basis for X^*. Notice that a bounded shrinking basis necessarily converges weakly to zero. The basis $\{x_n\}_{n=1}^{\infty}$ is *boundedly complete* provided that whenever the sequence $\{\sum_{i=1}^n \alpha_i x_i\}_{n=1}^\infty$ is bounded, then it is convergent. If a basis is shrinking, then its biorthogonal functionals are a boundedly complete basis for X^*. Conversely, if the biorthogonal functionals form a boundedly complete basis for their closed linear span Y, then the basis is shrinking (and hence $Y = X^*$). Similarly, a basis is boundedly complete if and only if its biorthogonal functionals are a shrinking basis for their closed linear span. If $\{x_n\}_{n=1}^\infty$ is a boundedly complete basis for X and Y is the normed closed linear span of the biorthogonal functionals $\{x_n^*\}_{n=1}^\infty$, the shrinkingness of $\{x_n^*\}_{n=1}^\infty$ implies that the natural evaluation mapping from X into Y^* is a surjective isomorphism (which is easily seen to be an isometry if the basis is monotone). Consequently, a space with a boundedly complete basis is isomorphic to a separable conjugate space. From these facts it follows, in particular, that **a Banach space X with a basis is reflexive if and only if some [or every] basis for X is both shrinking and boundedly complete.**

If X has an unconditional basis, then some [or every] unconditional basis is boundedly complete if and only if X has no subspace isomorphic to c_0, while some [or every] unconditional basis is shrinking if and only if no [or no complemented] subspace of X is isomorphic to ℓ_1. Consequently, **a space with unconditional basis is reflexive if and only if no subspace is isomorphic to either c_0 or to ℓ_1** if and only if its second dual is separable. In contrast, there is a nonreflexive space which has both a shrinking basis and a boundedly complete basis which is of codimension one in its bidual, [14, 1.d.2].

While most of the separable spaces encountered in classical analysis have bases, many do not have an unconditional basis; in particular, the spaces $L_1(0,1)$ and $C[0,1]$. Indeed, c_0 does not embed isomorphically into $L_1(0,1)$ (for example, because $L_1(0,1)$ is weakly sequentially complete and c_0 is not; see section 4), so any unconditional basis for $L_1(0,1)$ would have to be boundedly complete and $L_1(0,1)$ would be isomorphic to a separable conjugate space. But $L_1(0,1)$ does not even embed isomorphically into a separable conjugate space (since if $L_1(0,1) \subset X^*$ and for $t \in (0,1)$ x_t^* is a weak* cluster point of the sequence $n^{-1}1_{(t,t+1/n)}$, then it can be checked that there is an uncountable subset S of $(0,1)$ and $\delta > 0$ so that for all $t \neq s$ in S, $\|x_t^* - x_s^*\| > \delta$). A more sophisticated (but actually technically easier) argument shows even that $L_1(0,1)$ does not embed isomorphically into a space with unconditional basis; see [14, 1.d.1]. Since, as mentioned in section 4, every separable space embeds isometrically into $C[0,1]$, $C[0,1]$ also does not embed isomorphically into a space with unconditional basis.

It is a deep result that there exist infinite dimensional spaces which do not contain an unconditionally basic sequence (see [37]). The example uses Banach spaces whose norm is not defined explicitly by a formula but by an implicit procedure. This method of defining a Banach space (or a norm) is very useful in many contexts.

A basis $\{x_n\}_{n=1}^\infty$ is called *symmetric* provided every permutation of $\{x_n\}_{n=1}^\infty$ is equivalent to $\{x_n\}_{n=1}^\infty$. In particular, every permutation of $\{x_n\}_{n=1}^\infty$ is a basis, so a symmetric basis is unconditional. A basis $\{x_n\}_{n=1}^\infty$ is called *subsymmetric* provided it is unconditional and equivalent to each subsequence of itself. A symmetric basis is subsymmetric [14, 3.a.3]. It is evident that if a basis is unconditional, symmetric, or subsymmetric, then the same is true for the biorthogonal functionals (in their closed linear span).

The unit vector bases for ℓ_p, $1 \le p < \infty$, and c_0 are symmetric, while the L_p-normalization of the Haar system is not a symmetric basis for $L_p(0,1)$ if $p \ne 2$. In fact, $L_p(0,1)$ has no subsymmetric basis if $p \ne 2$. We have already mentioned that $L_1(0,1)$ does not have even an unconditional basis. That $L_p(0,1)$, $2 < p < \infty$, has no subsymmetric basis follows from the dichotomy principle for sequences in the space, which is discussed in section 8. This principle says that if $\{x_n\}_{n=1}^\infty$ is a seminormalized basic sequence in $L_p(0,1)$, $2 < p < \infty$, then $\{x_n\}_{n=1}^\infty$ has a subsequence which is equivalent to the unit vector basis for either ℓ_p or ℓ_2. Since, as noted in section 4, $L_p(0,1)$, $p \ne 2$, is not isomorphic to either ℓ_p or ℓ_2, $L_p(0,1)$, $2 < p < \infty$, has no subsymmetric basis. That $L_p(0,1)$, $1 < p < 2$, has no subsymmetric basis then follows by duality.

It is interesting that symmetric bases are nevertheless prevalent. For example, any space which has an unconditional basis is isomorphic to a complemented subspace of a space which has a symmetric basis (this will be discussed further in section 11). In fact, there is even a space Y which has a symmetric basis and also an unconditional basis $\{z_n\}_{n=1}^\infty$ which is *universal for unconditional bases* in the sense that every unconditional basis for any Banach space is equivalent to a subsequence of $\{z_n\}_{n=1}^\infty$! The space Y and the universal unconditional basis is easy to construct, given the fact, to be proved in section 4, that every separable Banach space is isometric to a subspace of $C[0,1]$. Indeed, one takes a dense sequence $\{x_n\}_{n=1}^\infty$ of non zero vectors in $C[0,1]$ and lets Y be the completion of the finitely supported scalar sequences under the norm $\|\{a_n\}_{n=1}^\infty\| := \sup_\pm \|\sum_n \pm a_n x_n\|_{C[0,1]}$. The unit vector basis in Y is a universal unconditional basis. See [14, p. 129] or [41] for further discussion and for the proof that the space Y has a symmetric basis.

While unconditional bases, symmetric bases, and subsymmetric bases are all useful strengthenings of the notion of basis, there are equally useful weakenings. A *Schauder decomposition* for a Banach space X is a sequence $\{X_n\}_{n=1}^\infty$ of nonzero closed subspaces of X such that every vector x in the space has a unique representation of the form $x = \sum x_n$ with $x_n \in X_n$. If each X_n is finite dimensional, the decomposition is called a *finite dimensional decomposition*, or simply an FDD. A basis can be regarded as an FDD $\{X_n\}_{n=1}^\infty$ such that every space X_n is one dimensional. As in the case of bases, a

Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ for a Banach space X determines a sequence $\{P_n\}_{n=1}^{\infty}$ (called the *partial sum projections* of the decomposition) of commuting projections with increasing ranges which converge strongly to the identity operator on X; namely, for $x = \sum x_n$ with $x_n \in X_n$ and $n = 1, 2, \ldots$, $P_n x = \sum_{i=1}^{n} x_i$. Conversely, if $\{P_n\}_{n=1}^{\infty}$ is a sequence of commuting projections with increasing ranges which converge strongly to the identity operator on X, then $\{(P_n - P_{n-1})X\}_{n=1}^{\infty}$ ($P_0 := 0$) is a Schauder decomposition for X for which $\{P_n\}_{n=1}^{\infty}$ is the sequence of partial sum projections. The biorthogonal functionals for a basis are replaced in the case of a Schauder decomposition by the sequence $\{(P_n^* - P_{n-1}^*)X^*\}_{n=1}^{\infty}$ of (even weak*) closed subspaces of X^*, which form a Schauder decomposition for their closed span. The supremum of the norms of the partial sum projections determined by a Schauder decomposition is finite and is called the *decomposition constant* of the decomposition, and the Schauder decomposition is called *monotone* if its decomposition constant is one.

The definitions of unconditional, shrinking, and boundedly complete bases generalize immediately to Schauder decompositions. The structure theory for bases goes over with little difficulty to FDD's. More importantly, there are useful concepts involving FDD's which are not merely generalizations of notions about bases. For example, a *blocking* of a Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ is a sequence of the form $\{X_{n_k} + \ldots + X_{n_{k+1}-1}\}_{k=1}^{\infty}$ with $1 = n_1 < n_2 < \ldots$. Any blocking of a basis (that is, of a Schauder decomposition into one dimensional spaces) produces an FDD which is not a basis. While bases and block bases are fine in situations where passage to subsequences or subspaces involves no loss, blockings of FDD's sometime provide the proper framework for investigations where it is important to maintain global control. For example, it was mentioned above that every normalized basic sequence in ℓ_p, $1 < p < \infty$, has a subsequence equivalent to the unit vector basis of ℓ_p, but one can prove that every FDD for a subspace of ℓ_p, $1 < p < \infty$, has a blocking which is an ℓ_p decomposition [14, 2.d.1]. (A Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ is said to be *an ℓ_p decomposition* provided that for $x_n \in X_n$, the series $\sum x_n$ converges if and only if $\sum \|x_n\|^p < \infty$).

A separable Banach space is said to have the *bounded approximation property* or simply BAP provided there is a sequence $\{T_n\}_{n=1}^{\infty}$ of finite rank operators on X which converges strongly to the identity. If the operators can be chosen to commute, the space has the *commuting bounded approximation property* (CBAP). In either case the supremum of $\|T_n\|$ is finite; if at most λ, we say that the space has the λ-BAP or λ-CBAP, or when $\lambda = 1$ the *metric approximation property* (MAP) or *commuting metric approximation property* (CMAP). Finally, A Banach space X has the *approximation property* or simply AP provided that for every compact subset K of X and $\epsilon > 0$, there is a finite rank operator T on X such that $\|x - Tx\| < \epsilon$ for every $x \in K$. If a space X has the AP, then every compact operator S into X is the norm limit of a sequence of finite rank operators. Indeed, take for $n = 1, 2, \dots$ a finite rank operator T_n on X so that for each x in the image under of the unit ball of the domain of S the inequality $\|x - T_n x\| \le 1/n$ holds. Then $\|S - T_n S\| \le 1/n$. The converse is also true [14, 1.e.4].

The implications basis ⇒ FDD ⇒ CBAP ⇒ BAP ⇒ AP are all easy or obvious. None are reversible except possibly CBAP ⇒ BAP; this is discussed in [24], as is the existence of spaces which fail the AP and connections among these various approximation conditions. There are some surprises, such as the AP implies the MAP in reflexive spaces; the MAP implies CMAP; and a space which has the BAP is isomorphic to a complemented subspace of a space which has a basis. Incidentally, while spaces failing the AP or other approximation properties might be thought pathological, they arise naturally in various contexts (for example, the nonseparable space $B(\ell_2)$ of operators on ℓ_2 fails the AP, as do some separable C^* algebras). Moreover, the construction of spaces failing approximation conditions leads to nice theorems which do not involve any notion of approximation; in particular, isomorphic characterizations of Hilbert space (see [35]).

The existence of spaces failing the AP as well as other considerations lead to the study of other basis-like structures in Banach spaces. Probably the most useful of these is that of Markuschevich basis. A *Markuschevich basis* for a Banach space X is a biorthogonal system $\{x_\gamma, x^*_\gamma\}_{\gamma\in\Gamma}$ for which $\{x_\gamma\}_{\gamma\in\Gamma}$ is *fundamental* (that is, the span of the x_γ's is dense in X) and $\{x^*_\gamma\}_{\gamma\in\Gamma}$ is *total*; that is, the x^*_γ's separate the points of X. The trigonometric system in $L_1(\mathbb{T})$ or $C(\mathbb{T})$ is a natural example of a Markuschevich basis which is not a (Schauder) basis in any order (but this last statement is not easy to verify). Markuschevich bases do not always exist in the nonseparable setting, but are a useful tool for investigating nonseparable spaces in situations where they exist (see [44]). It is easy to see that any separable space X admits a Markuschevich basis. One starts with linearly independent sequences $\{y_n\}_{n=1}^\infty$ in X and $\{y^*_n\}_{n=1}^\infty$ in X^* with $\{y_n\}_{n=1}^\infty$ fundamental and $\{y^*_n\}_{n=1}^\infty$ total and applies a Gram-Schmidt type procedure to biorthogonalize the sequences (alternating between working in X and X^*) to produce a Markuschevich basis $\{x_n, x^*_n\}_{n=1}^\infty$ so that $\operatorname{span}\{x_n\}_{n=1}^\infty = \operatorname{span}\{y_n\}_{n=1}^\infty$ and $\operatorname{span}\{x^*_n\}_{n=1}^\infty = \operatorname{span}\{y^*_n\}_{n=1}^\infty$ (see [14, 1.f.3]). A deeper fact ([14, 1.f.4]) is that a separable space contains a Markuschevich basis $\{x_n, x^*_n\}_{n=1}^\infty$ for which $\sup_n \|x_n\|\|x^*_n\| \le 20$, and it is known that twenty can be replaced by any number larger than one. We shall see in section 8 that a finite dimensional space has a basis $\{x_n, x^*_n\}_{n=1}^N$ for which $\|x_n\|\|x^*_n\| = 1$ for each n, but it is an open problem whether every separable space has a Markuschevich basis with this property. One separable theorem in which Markuschevich bases are used but do not appear in the statement is that every separable space X has a subspace Y such that both Y and X/Y have an FDD ([14, 1.g.2]). Incidentally, it is open whether the conclusion can be improved to "both Y and X/Y have a basis".

4 Classical spaces

In this section we present some basic facts concerning the structure of the classical spaces $C(K)$ and $L_p(\mu)$, their classification, and the relations among the spaces. Most books on real analysis, such as [18], contain some elementary results about the structure of these spaces. Beyond the most basic material, [21] is accessible to students and has the additional advantage that some of the relations between these spaces and other spaces encountered in analysis are touched on. The more narrowly focused book [7] is also directed at students, contains several structural results omitted from [21], and the author's off-the-wall style makes the book fun to read.

A normalized sequence $\{x_n\}_{n=1}^{\infty}$ of disjointly supported functions in $L_p(\mu)$, $1 \le p < \infty$, is 1-equivalent (see section 3) to the unit vector basis for ℓ_p and the closed span of $\{x_n\}_{n=1}^{\infty}$ is the range of a norm one projection. The first statement is clear and the projection is defined by $x \mapsto \sum_n \left(\int x|x_n|^{p-1}\text{sign}\, x_n \,\mathrm{d}\mu\right) x_n$. In particular, every block basis of the unit vector basis for ℓ_p, $1 \le p < \infty$, spans a space isometric to ℓ_p onto which there is a norm one projection. This last statement is also true if ℓ_p is replaced by c_0. Hence by the principle of small perturbations discussed in section 3, if Y is an infinite dimensional subspace of X, $X = \ell_p$ for $1 \le p < \infty$ or $X = c_0$, then for any $\epsilon > 0$ there is a subspace Z of Y with $d(Z, X) < 1 + \epsilon$ and so that there is a projection of norm less than $1 + \epsilon$ from X onto Z.

Although general subspaces of X, $X = \ell_p$ for $1 \le p \ne 2 < \infty$ or $X = c_0$ can be bad in that they can fail the approximation property and there are many of them, in a sense that is made precise in [24], the isomorphism theory of complemented subspaces of X is simple: **An infinite dimensional complemented subspace Y of X, $X = \ell_p$ for $1 \le p < \infty$ or c_0, is isomorphic to X**. To see this, write $X \approx Y \oplus V$ for some V. As mentioned in the previous paragraph, also $Y \approx X \oplus W$ for some W. Hence (we write the formula for ℓ_p; for c_0 the proof is identical but the notation changes)

$$\begin{aligned}\ell_p &\approx (\ell_p \oplus \ell_p \oplus \cdots)_p \approx (Y \oplus Y \oplus \cdots)_p \oplus (V \oplus V \oplus \cdots)_p \\ &\approx Y \oplus (Y \oplus Y \oplus \cdots)_p \oplus (V \oplus V \oplus \cdots)_p \approx Y \oplus \ell_p \approx W \oplus \ell_p \oplus \ell_p \approx Y.\end{aligned}$$

This method of proof, which is useful in many contexts, is called the *decomposition method.*

It is also true that every infinite dimensional complemented subspace of ℓ_∞ is isomorphic to ℓ_∞ but the proof requires an additional argument ([14, 2.a.7]).

Let $1 \le p < r < \infty$. In section 3 we saw that every operator from a subspace of ℓ_r into ℓ_p is compact. There are noncompact operators from ℓ_p into ℓ_r, such as the formal identity operator $I_{p,r}$. When there exist "nonsmall" operators from one space to another (here "nonsmall" means "noncompact", but elsewhere it can have another meaning), it is helpful to have one "nonsmall" operator which is typical of the genre. In fact, $I_{p,r}$ is typical of the noncompact operators from ℓ_p into ℓ_r in a sense that can be made precise:

Say that an operator $T : X \to Y$ *factors through* an operator $S : W \to Z$ provided there exist operators $U : X \to W$ and $V : Z \to Y$ so that the diagram

$$\begin{array}{ccc} W & \xrightarrow{S} & Z \\ U \uparrow & & \downarrow V \\ X & \xrightarrow{T} & Y \end{array}$$

commutes; that is, $T = VSU$. Now we can make precise the statement about $I_{p,r}$: **If T is a noncompact operator from a subspace of ℓ_p into ℓ_r, $1 \le p < r < \infty$, then $I_{p,r}$ factors through T.** Indeed, this is an immediate consequence of the comments in section 3 on noncompact operators from a subspace of a space with a basis and the remark in this section that a block basic sequence of the unit vector basis of ℓ_p spans a complemented copy of ℓ_p in ℓ_p.

It is easily seen that the only separable $L_\infty(\mu)$ spaces are those for which the measure μ is purely atomic and has finitely many, say n, atoms, in which case $L_\infty(\mu)$ is isometric to ℓ_∞^n since it is obvious that the indicator functions of n distinct atoms are, in $L_\infty(\mu)$, 1-equivalent to the unit vector basis for ℓ_∞^n.

The isometric classification of the separable $L_p(\mu)$-spaces, $1 \le p < \infty$, is also simple. First, the measure μ can be assumed to be σ-finite. Secondly, any $L_p(\mu)$, $1 \le p \le \infty$ with μ σ-finite, is isometric to a space of the same form with μ replaced by a probability measure. Indeed, take g positive a.e. $[\mu]$ with integral one and define a measure ν by $d\nu = g d\mu$. Then the mapping $f \mapsto f \cdot g^{-1/p}$ defines an isometry from $L_p(\mu)$ onto $L_p(\nu)$. (For $p = \infty$ the isometry is just the formal identity).

Next, if $L_p(\mu)$ is separable with μ a purely nonatomic probability measure, then the isomorphism theorem for separable measure algebras ([18, p. 399]) yields that $L_p(\mu)$ is isometric to $L_p(0,1)$. On the other hand, if the measure μ is purely atomic with atoms $\{A_\gamma\}_{\gamma \in \Gamma}$, then the function which maps for each γ the unit vector e_γ to $\mu(A_\gamma)^{-1/p}$ times the indicator function of A_γ extends to an isometry from $\ell_p(\Gamma)$ onto $L_p(\mu)$. Now if x and y are in $L_p(\mu)$, $1 \le p \ne 2 < \infty$, then x and y are disjointly supported if and only if $\|x \pm y\|_p^p = \|x\|_p^p + \|y\|_p^p$ ([18, p. 416]). This implies that an isometry from one L_p space onto another preserves disjointness and therefore also atoms. All these remarks combine to show that ℓ_p^n, ℓ_p, $L_p(0,1)$, $\ell_p \oplus_p L_p(0,1)$, $\ell_p^n \oplus_p L_p(0,1)$, $n = 1, 2, \ldots$ is a complete listing, up to isometry, of the separable $L_p(\mu)$ spaces when $1 \le p \ne 2 < \infty$, and these are all mutually nonisometric. Of course, in the Hilbertian case $p = 2$, ℓ_2^n, $n = 1, 2, \ldots$; ℓ_2 is the appropriate listing.

The decomposition method then yields that ℓ_p^n, ℓ_p, $L_p(0,1)$ is a complete listing, up to isomorphism, of the separable L_p spaces. Later in this section it is noted that ℓ_p is not isomorphic to $L_p(0,1)$ when $1 \le p \ne 2 < \infty$, so this is a listing of nonisomorphic spaces for these values of p.

In order to study the structure of subspaces of $L_p(0,1)$ as well as to investigate many other questions about L_p for one fixed value of p, it is convenient to consider the scale of L_p spaces as p varies. One reason for this is that (as we shall see) there are spaces of functions X on $(0,1)$ on which the norm $\|\cdot\|_{p_1}$ is equivalent to $\|\cdot\|_{p_2}$ with $p_1 < p_2$. Note that when this occurs, since $\|\cdot\|_p \le \|\cdot\|_r$ when $p \le r$, all the norms $\|\cdot\|_p$ are equivalent on X for $p_1 \le p \le p_2$. In fact, the *extrapolation principle* says that then all the norms $\|\cdot\|_p$ are equivalent on X for $p \le p_2$. Indeed, suppose C is a constant so that $\|x\|_{p_2} \le C\|x\|_{p_1}$ for $x \in X$, $0 < p < p_1$, and $0 < \lambda < 1$ is defined by the formula $p_1 = \lambda p + (1-\lambda)p_2$. Then, by Hölder's inequality, for $x \in X$ we have

$$\|x\|_{p_1} \le \|x\|_p^{\lambda} \|x\|_{p_2}^{(1-\lambda)} \le C^{1-\lambda} \|x\|_{p_1}^{1-\lambda} \|x\|_p^{\lambda}$$

and hence $C^{1-1/\lambda}\|x\|_{p_1} \le \|x\|_p \le \|x\|_{p_1}$.

The preceding yields particularly good information in case $2 < p$ and X is a closed subspace of $L_p(0,1)$ which is closed in $L_r(0,1)$ for some $r \ne p$. By the open mapping theorem, the $\|\cdot\|_p$ and $\|\cdot\|_r$ norms are equivalent on X, and hence so is the $\|\cdot\|_2$ norm. This means that X must be isomorphic to a Hilbert space. Moreover, the orthogonal projection P from $L_2(0,1)$ onto X induces a bounded linear projection $\tilde{P}$ from $L_p(0,1)$ onto X. The extrapolation principle then says that $\|\cdot\|_{p^*}$, $1/p + 1/p^* = 1$, and $\|\cdot\|_2$ are equivalent on X. The orthogonal projection onto X extends to the bounded projection $\tilde{P}^*$ from $L_{p^*}(0,1)$ onto X.

Suppose that, on the other hand, X is a closed subspace of $L_p(0,1)$, $1 \le p < \infty$, and for some (or, equivalently, every) $0 < r < p$ the $\|\cdot\|_p$ and $\|\cdot\|_r$ norms are not equivalent on X. Hölder's inequality then yields that for any $M < \infty$ the infimum of $\|x 1_{[|x| \le M]}\|_p$ as x ranges over the the unit sphere of X is zero. This means that X contains unit vectors which are an arbitrarily small perturbation of vectors whose supports have arbitrarily small measure, which in turn yields that X contains a sequence of unit vectors which is an arbitrarily small perturbation of a sequence of disjointly supported vectors in $L_p(0,1)$ and hence, by the principle of small perturbations, that X contains a subspace which is isomorphic to ℓ_p and complemented in $L_p(0,1)$.

These observations yield a dichotomy principle for subspaces of $L_p(0,1)$, $2 < p < \infty$ [21, III.A.4]. **Let X be a closed subspace of $L_p(0,1)$, $2 < p < \infty$. Then either X is isomorphic to a Hilbert space and complemented in $L_p(0,1)$ or, for each $C > 1$, X contains a subspace C-isomorphic to ℓ_p and C-complemented in $L_p(0,1)$.** This is stated for $L_p(0,1)$, but is valid for general $L_p(\mu)$ by the isometric classification of L_p spaces discussed earlier. Notice also that the comments in section 3 about ℓ_p now yield that if ℓ_r embeds isomorphically into $L_p(\mu)$, $0 < r < \infty$ and $2 < p < \infty$, then either $r = p$ or $r = 2$.

The natural examples of infinite dimensional function spaces which are closed in L_p for more than one value of p come from probability theory. Suppose that $\{g_n\}_{n=1}^{\infty}$ is a sequence of independent standard Gaussian random variables, discussed in section 2. From the form of the distribution of g it is evident that $\|g\|_p$ is finite for all finite p and $\{g_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $L_2(0,1)$. A characteristic function argument yields that if $\sum_{k=1}^{n} |\alpha_k|^2 = 1$, then $\sum_{k=1}^{n} \alpha_k g_k$ is again a standard Gaussian and hence has the same norm in $L_p(0,1)$ as g. This means that for all $0 < p < \infty$, the mapping $e_n \mapsto \|g\|_p^{-1} g_n$, $n = 1, 2, \ldots$, extends to an isometry from ℓ_2 onto a subspace X of $L_p(0,1)$ where X does not even depend on p, and the orthogonal projection onto X defines a bounded projection from $L_p(0,1)$ onto X if $1 < p < \infty$ (in section 10 we explain why no isomorph of ℓ_2 in $L_1(0,1)$ or $L_\infty(0,1)$ can be complemented). In particular, the dichotomy principle for subspaces of L_p, $2 < p < \infty$, is not a monochotomy principle. This also implies, in view of the structure theory of ℓ_p discussed earlier, that ℓ_p and $L_p(0,1)$ are not isomorphic for $0 < p \neq 2 < \infty$.

A second natural example from probability theory of a subspace of L_p, $0 < p < \infty$, which is isomorphic (but not isometric for $p \neq 2$) to ℓ_2 comes from considering a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of independent random variables each taking on each of the values 1 and -1 with probability 1/2. Such a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ is called a *Rademacher sequence.* Classically the *Rademacher functions* $\{r_n\}_{n=1}^{\infty}$ are the concrete realization of such a sequence on $[0,1]$ defined by $r_n := \sum_{k=0}^{2^n-1} h_{2^n+k}$, where $\{h_n\}_{n=1}^{\infty}$ is the Haar system. *Khintchine's inequality* says that a Rademacher sequence is equivalent, in the L_p norm, $0 < p < \infty$, to the unit vector basis of ℓ_2; for future reference we write the inequality explicitly with the best constants labeled A_p and B_p:

$$A_p \left(\sum |\alpha_n|^2\right)^{1/2} \leq \left(\mathbb{E}|\sum \alpha_n \varepsilon_n|^p\right)^{1/p} \leq B_p \left(\sum |\alpha_n|^2\right)^{1/2}. \qquad (1)$$

Since a Rademacher sequence is orthonormal, $(\sum |\alpha_n|^2)^{1/2} = \|\sum \alpha_n \varepsilon_n\|_2 \equiv (\mathbb{E}|\sum \alpha_n \varepsilon_n|^2)^{1/2}$, it follows that $A_p = 1$ for $p \geq 2$ and $B_p = 1$ for $p \leq 2$, and the exact values of A_p and B_p are known. The most elementary proof of Khintchine's inequality proceeds by checking the right inequality in (1) for p an even integer. This clearly gives the result for all $2 \leq p < \infty$ and one then obtains the result for $0 < p < 2$ from the extrapolation principle. The less computational modern proof of Khintchine's inequality gives a vector valued version of Khintchine's inequality called the *Kahane-Khintchine inequality.* This will be done in section 8.

The existence of an r-stable variable g (see section 2) for $0 < r < 2$ shows that the subspace structure of $L_p(0,1)$, $0 < p < 2$, is much more complicated than that of $L_p(0,1)$, $2 < p < \infty$. Indeed, just as in the Gaussian case, there exists for $0 < r < 2$ a sequence $\{g_n\}_{n=1}^\infty$ of symmetric r-stable random variables defined on $(0,1)$. If $0 < p < r$, then these random variables are in $L_p(0,1)$. Now if $\sum_{k=1}^n |\alpha_k|^r = 1$, then $\sum_{k=1}^n \alpha_k g_k$ is again symmetric r-stable and hence has the same norm in $L_p(0,1)$ as g. This means that for all $0 < p < r$, the mapping $e_n \mapsto \|g\|_p^{-1} g_n$, $n = 1, 2, \ldots$, extends to an isometry from ℓ_r into $L_p(0,1)$. In section 9 we explain how to derive from this the fact that for $1 \leq p < r < 2$, $L_r(0,1)$ embeds isometrically into $L_p(0,1)$. It turns out that this covers all the cases in which ℓ_r embeds isomorphically into $L_p(\mu)$ with p and r finite. The remaining cases are discussed in section 8.

In the reflexive range $1 < p < \infty$, information about subspaces of $L_{p^*}(\mu)$, $1/p + 1/p^* = 1$, given above gives information on quotients of $L_p(\mu)$. It turns out that the case $p = 1$ is quite different: **Every separable Banach space X is isometric to a quotient of ℓ_1.** Indeed, if $\{x_n\}_{n=1}^\infty$ is dense in the unit ball of X, the linear extension of the map $e_n \mapsto x_n$ (where $\{e_n\}_{n=1}^\infty$ is the unit vector basis for ℓ_1) maps the unit ball of ℓ_1 onto a dense subset of the unit ball of X and hence extends to a quotient mapping from ℓ_1 onto X. Similarly, if the Banach space X has density character κ, then X is isometric to a quotient of $\ell_1(\Gamma)$ when Γ has cardinality κ.

Another useful and interesting property of ℓ_1 (which also holds for $\ell_1(\Gamma)$ for any set Γ) is the *lifting property:* If T is an operator from a Banach space X onto ℓ_1 then there is a *lifting* S of T; that is, an operator $S : \ell_1 \to X$ for which $TS = I_{\ell_1}$. Indeed, by the open mapping theorem there are x_n in X with $Tx_n = e_n$ and $\lambda := \sup \|x_n\| < \infty$. The mapping $e_n \mapsto x_n$ then extends to an operator $S : \ell_1 \to X$ with $\|S\| = \lambda$ satisfying $TS = I_{\ell_1}$.

As noted in section 3, ℓ_1 has the Schur property; that is, weakly convergent sequences in ℓ_1 are norm convergent. An immediate consequence is that every weakly compact subset of ℓ_1 is norm compact. The weakly compact subsets of $L_1(0,1)$ are more complicated since $L_1(0,1)$ contains infinite dimensional reflexive subspaces. There is however a nice characterization of subsets of $L_1(\mu)$

which have weakly compact closure when μ is a finite measure. First, if X is any Banach space and W is a subset of X such that for each $\epsilon > 0$ there exists a weakly compact set S so that $W \subset S + \epsilon B_X$, then W has weakly compact closure (use the fact that a bounded subset of X is weakly compact if its weak* closure in X^{**} is a subset of X). Next, given $W \subset L_1(\mu)$ with μ a finite measure, set for $k \in \mathbb{N}$, $a_k = a_k(W) := \sup\{\|x 1_{|x|\geq k}\|_1 : x \in W\}$. Clearly $\{a_k\}_{k=1}^\infty$ decreases to some $a = a(W) \geq 0$. The set W is called *uniformly integrable* if $a = 0$. If $a = 0$ then W has weakly compact closure because W is a subset of $kB_{L_\infty(\mu)} + a_k B_{L_1(\mu)}$ and $B_{L_\infty(\mu)}$ is weakly compact in $L_1(\mu)$.

If $a(W) > 0$, the set W does not have weakly compact closure and in fact even contains a sequence equivalent to the unit vector basis of ℓ_1. To see this, define two further numerical parameters for a subset W of $L_1(\mu)$; namely, $b(W) := \sup \lim_{n\to\infty}\{\|x_n 1_{A_n}\|_1\}$, where the supremum is over all sequences $\{x_n\}_{n=1}^\infty$ in W and $\{A_n\}_{n=1}^\infty$ of measurable sets with $\mu A_n \to 0$; and $c(W)$, defined the same way as $b(W)$ except that the supremum is over all sequences of disjoint measurable sets. It is an elementary exercise in measure theory to verify that $a(W) = b(W) = c(W)$. Now if $c(W) > 0$, take a sequence $\{x_n\}_{n=1}^\infty$ in W and a sequence $\{A_n\}_{n=1}^\infty$ of disjoint measurable sets so that $0 < \|x_n 1_{A_n}\|_1 \to c(W)$. Clearly $c(\{x_n 1_{\tilde{A_n}}\}) = 0$, so $\{x_n 1_{\tilde{A_n}}\}$ has weakly compact closure. The sequence $\{x_n 1_{A_n}\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1 since the A_n's are disjoint and $\|x_n 1_{A_n}\|$ is bounded away from zero. This implies that a subsequence of $\{x_n\}_{n=1}^\infty$ is also equivalent to the unit vector basis of ℓ_1. (Remark: If $\{y_n\}_{n=1}^\infty$ is bounded in any $L_1(\mu)$ space and there is a sequence $\{B_n\}_{n=1}^\infty$ of disjoint measurable sets and $b > 0$ so that for all n, $\sum_{k\neq n}\|y_n 1_{B_k}\|_1 \leq b/2 < b \leq \|y_n 1_{B_n}\|_1$, then $\{y_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1. Moreover, the closed span Y of $\{y_n\}_{n=1}^\infty$ is complemented in $L_1(\mu)$. Indeed, set $B := \bigcup_{n=1}^\infty B_n$. The sequence $\{y_n 1_B\}_{n=1}^\infty$ is a small perturbation of the disjoint sequence $\{y_n 1_{B_n}\}_{n=1}^\infty$ and hence there is a projection P from $L_1(\mu)$ onto the closed span Z of $\{y_n 1_B\}_{n=1}^\infty$. The mapping $y_n 1_B \mapsto y_n$ extends to an operator T from Z into the closed span of $\{y_n\}_{n=1}^\infty$ since $\{y_n 1_B\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1. Then $f \mapsto TP(f1_B)$ defines a projection onto Y.) To apply this remark one uses a combinatorial argument to verify that there ae $n_1 < n_2 < \ldots$ so that $y_k := x_{n_k}$, $B_k := A_{n_k}$ satisfy the above conditions for any $0 < b < c(W)$.

It follows from the observations made in the beginning of this section that any separable subspace of any $L_1(\mu)$ space embeds isometrically into $L_1(0,1)$. Consequently, **a bounded subset of $L_1(\mu)$ either has weakly compact closure or contains a sequence equivalent to the unit vector basis of ℓ_1.** From this dichotomy principle and the fact that a weakly compact subset of a Banach space is weakly sequentially compact it follows that $L_1(\mu)$ **is weakly sequentially complete.** It also follows that **a nonreflexive subspace of $L_1(\mu)$ contains a subspace which is isomorphic to ℓ_1 and is**

complemented in $L_1(\mu)$.

There is a dichotomy principle, called *Rosenthal's ℓ_1 theorem,* for general Banach spaces which generalizes the one above for L_1: **A bounded sequence in a Banach space either has a weakly Cauchy subsequence or a subsequence which is equivalent to the unit vector basis of ℓ_1.** See [30] or [14, 2.5.e] or [7, Ch. XI].

A space Z is called λ-*injective* provided that for all Banach spaces X, every operator T from every subspace of X into Z has an extension to an operator $\tilde{T}$ from X into Z for which $\|\tilde{T}\| \leq \lambda\|T\|$. The spaces $\ell_\infty(\Gamma)$ are 1-injective. This is proved by composing an operator T into Z with each of the evaluation functionals at points of Γ and applying the usual Hahn-Banach theorem. More generally, $L_\infty(\mu)$ is 1-injective if μ is σ-finite. That is because $L_\infty(\mu)$ is a complete lattice (see section 5) in this case and the usual proof of the scalar Hahn-Banach theorem carries over.

It is obvious that ℓ_∞ embeds isometrically into $L_\infty(0,1)$, and $L_\infty(0,1)$ embeds isometrically into ℓ_∞ because its predual, $L_1(0,1)$, is a quotient of ℓ_1, the predual of ℓ_∞. Since both ℓ_∞ and $L_\infty(0,1)$ are 1-injective, each space is isomorphic to a complemented subspace of the other. A simplified version of the decomposition method then shows that ℓ_∞ and $L_\infty(0,1)$ are isomorphic. Surprisingly, there is no known operator which exhibits explicitly the isomorphism between these two concrete spaces.

The space c_0 has a weaker injectivity property: it is **separably 2-injective,** meaning that for all Banach spaces X and every operator T from a subspace Y of X with X/Y separable, there is an extension of T to an operator $\tilde{T}$ from X into c_0 for which $\|\tilde{T}\| \leq 2\|T\|$. Indeed, since ℓ_∞ is 1-injective, T extends to an operator $\tilde{T}$ from X into ℓ_∞ having the same norm as T and the range of $\tilde{T}$ is separable because X/Y is separable. Thus it is enough to check that if Z is a separable space which contains c_0, then there is a projection of norm at most two from Z onto c_0. To see this, first let z_n^* be an extension of the nth unit vector in $\ell_1 = c_0^*$ to a norm one element of Z^*. Let $d(\cdot,\cdot)$ be a translation invariant metric on Z^* which induces the weak* topology on the unit ball of Z^*; for example, if $\{x_n\}_{n=1}^\infty$ is dense in the unit sphere of Z the metric can be defined by $d(x^*,y^*) = \sum_{n=1}^\infty 2^{-n}|(x^*-y^*)(x_n)|$. Since every weak* limit point of $\{z_n^*\}_{n=1}^\infty$ belongs to the unit ball B of the annihilator of c_0 in Z^*, it follows that $d(z_n^*, B) \to 0$. Thus we can choose w_n^* in B so that $d(z_n^*, w_n^*) \to 0$, which means that $z_n^* - w_n^* \to 0$ weak*. The formula $Pz := \{(z_n^* - w_n^*)(z)\}_{n=1}^\infty$ defines a projection of norm at most two from Z onto c_0.

That the space c_0 is not separably λ-injective for any $\lambda < 2$ can be seen by considering $P1$ for any projection P from c onto c_0. More interesting is that the separability assumption is needed: **There is no projection from ℓ_∞**

onto c_0**.** Indeed, otherwise we would have $\ell_\infty \sim c_0 \oplus \ell_\infty/c_0$. But ℓ_∞/c_0 is not isomorphic to a subspace of ℓ_∞. This can be seen by proving that $c_0(\mathbb{R})$ embeds into ℓ_∞/c_0 but not into ℓ_∞. $c_0(\mathbb{R})$ admits no countable separating family of linear functionals, so it does not embed into ℓ_∞. To see that $c_0(\mathbb{R})$ embeds into ℓ_∞/c_0, take a family $\{A_r\}_{r\in\mathbb{R}}$ of infinite subsets of $\mathbb{N}$ so that the intersection of any two is finite (replace $\mathbb{N}$ by the rationals and for each $r \in \mathbb{R}$ consider a sequence of rationals which converges to r). Let x_r be the image in the quotient space ℓ_∞/c_0 of the indicator function of A_r. It is easy to check that $\{x_r\}_{r\in\mathbb{R}}$ is 1-equivalent to the unit vector basis of $c_0(\mathbb{R})$.

The space c_0 is the only (up to isomorphism) separable space which is separably injective [43]. General $C(K)$ spaces do have another useful into extension property: **A compact operator T from a subspace Y of a Banach space X into $C(K)$ has an extension $\tilde{T}$ to a compact operator from X into $C(K)$.** Indeed, since $C(K)$ spaces have the BAP, T can be approximated in the operator norm by operators of finite rank and hence we can write $T = \sum_{n=0}^{\infty} T_n$ with each T_n of finite rank and $\|T_n\| < 2^{-n}$ for $n \geq 1$. Therefore it is enough to observe that if $S : X \to C(K)$ has finite rank, then S extends to a finite rank operator $\tilde{S} : X \to C(K)$ with $\|\tilde{S}\| \leq (1+\epsilon)\|S\|$ (where $\epsilon > 0$ is arbitrary). But by using partitions of unity and the principle of small perturbations one checks that there is a subspace E of $C(K)$ so that $SX \subset E$ and $d(E, \ell_\infty^n) < 1 + \epsilon$, where $n = \dim E < \infty$. Then E is $(1 + \epsilon)$-injective and hence the desired extension of S exists. The same argument works when $C(K)$ is replaced by any $\mathcal{L}_\infty$ space (defined in section 9).

Every $L_\infty(\mu)$ is isometric to $C(K)$ for some compact Hausdorff space K. This follows from Gelfand theory, since $L_\infty(\mu)$ is a commutative B^* algebra with unit (see [19, Ch. 11]). Alternatively, it follows from lattice characterizations of $C(K)$ spaces (see section 5).

The $C(K)$ spaces play a special rôle in Banach space theory because they are a *universal class:* **Every Banach space X is isometric to a subspace of some $C(K)$ space.** This can be seen by embedding X into $\ell_\infty(\Gamma)$ for Γ appropriately large and applying the comment in the previous paragraph. Alternatively, X isometrically embeds via evaluation into $C(K)$ with K the unit ball of X^* with the weak* topology. This approach is preferable because the unit ball of X^* is weak* metrizable when X is separable. In fact, **every separable Banach space X embeds isometrically into $C[0,1]$.** First, $C(\Delta)$ embeds isometrically into $C[0,1]$, where Δ is the usual "middle thirds" Cantor set in $[0,1]$, by extending a continuous function on Δ affinely on the component intervals of the complement of Δ in $[0,1]$. Next, using that Δ is homeomorphic to $\{0,1\}^{\mathbb{N}}$, it can be shown that every compact metric space K is the continuous image of Δ. Build a tree structure $K(i_1, i_2, \ldots, i_n) : \{i_1, i_2, \ldots, i_n\} \in \{0,1\}^n; n = 1, 2, \ldots\}$ of nonempty closed subsets of K so that each $K(i_1, i_2, \ldots, i_n)$ is the union of $K(i_1, i_2, \ldots, i_n, 0)$ and $K(i_1, i_2, \ldots, i_n, 1)$, $K = K_0 \cup K_1$, and the maximum diameter of $K(i_1, i_2, \ldots, i_n)$ as $\{i_1, i_2, \ldots, i_n\}$ varies over $\{0,1\}^n$ tends to zero as $n \to \infty$. This is not hard to do using compactness of K. The map defined by $\{i_j\}_{j=1}^{\infty} \mapsto K(i_1) \cap K(i_1, i_2) \cap K(i_1, i_2, i_3) \cap \ldots$ is then a continuous surjection from $\Delta = \{0,1\}^{\mathbb{N}}$ onto K. Having gotten a continuous mapping g from Δ onto K, one embeds $C(K)$ into $C(\Delta)$ by the formula $Tx = x \circ g$, $x \in C(K)$.

The isometric classification of $C(K)$ spaces is easy and can be found in beginning texts (such as [12, Th. 231]): **The space $C(K)$ is isometric to $C(H)$ if and only if K is homeomorphic to H.** This follows from the identification of the extreme points of the unit ball of $C(K)^*$ as the evaluation functionals at points of K (as well as multiples of these functionals by scalars of magnitude one). The much harder problem of the isomorphic classification of $C(K)$ spaces has been accomplished in the separable case (that is, for compact metric spaces): **If K is an uncountable compact metric space then $C(K)$ is isomorphic to $C(0,1)$.** If K is a countable compact metric space then K is homeomorphic to the space $[1, \alpha]$ of all ordinals up to the ordinal α for some countable ordinal α in the order topology. $C(1, \alpha)$ **is isomorphic to** $C(1, \beta)$ **when $\alpha < \beta$ if and only if $\beta < \alpha^{\omega}$.** See [40] for further discussion of the separable case. It seems a hopeless task to get an isomorphic classification of general $C(K)$ spaces, but some information is contained in [44].

We have already mentioned inexplicitly that the dual of an $L_1(\mu)$ space is isometric to a $C(K)$ space. Similarly, the dual of a $C(K)$ is isometric to $L_1(\mu)$ for some measure μ. The usual representation (see [18]) of the dual of $C(K)$ is the space $M(K)$ of finite signed measures on the sigma algebra of Baire subsets of K (that is, the sigma algebra generated by the closed G_δ subsets of K). It is a routine exercise using the Radon-Nikodým theorem to verify that if $\{\mu_\gamma\}_{\gamma \in \Gamma}$ is a maximal family of mutually singular Baire probability measures on K, then $M(K)$ is isometric to $\left(\sum_{\gamma \in \Gamma} L_1(\mu_\gamma)\right)_1$.

Once the duals of $C(K)$ and of $L_1(\mu)$ are classified, it is natural to ask what are the preduals of $C(K)$ and $L_1(\mu)$? It turns out that every *isometric* predual of a space $C(K)$ is isometric to $L_1(\mu)$ for some measure μ and that all preduals of $C(K)$ are mutually isometric. On the other hand, the *isomorphic* preduals of, for example, ℓ_∞, form a rich class of spaces which are still not well understood. Preduals of $L_1(\mu)$ spaces are even less well understood. Here we mention only that the space ℓ_1 has uncountably many mutually nonisomorphic preduals among the $C(K)$ spaces ($C(K)^*$ is isometric to ℓ_1 if K is a countable compact space).

5 Banach lattices

Most of the Banach spaces that appear naturally in analysis carry structure in addition to their structure as Banach spaces. It turns out that this additional structure can affect properties which are defined purely in terms of Banach space concepts (such as linear operators and duality). An extra structure that classical spaces such as L_p and $C(K)$ spaces possess is that of a *Banach lattice*, which is a Banach space over the reals that is equipped with a partial order $\leq$ for which $x \vee y$ and $x \wedge y$ exist for all vectors x, y, and such that the positive cone is closed under addition and multiplication by nonnegative real numbers and the order is connected to the norm by the condition that $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$, where the absolute value is defined by $|x| = x \vee (-x)$.

A linear mapping from a Banach lattice to a Banach lattice is *positive* if it carries positive vectors to positive vectors. This is equivalent to saying that the mapping is order preserving. It is easy to see that a positive linear mapping is continuous, so we call them *positive operators.* If a positive operator preserves the lattice operations, it is called a *lattice homomorphism.* It is clear that a one-to-one surjective positive operator T whose inverse is positive is a *lattice isomorphism;* that is, both T and T^{-1} are lattice homomorphisms. The dual of a Banach lattice X is again a Banach lattice under the standard ordering on X^* in which the positive linear functionals form the positive cone. With this definition it is easy to see that the canonical mapping from X into X^{**} is positive ([15, 1.a.2]).

While there are interesting functional analytical topics (such as positive operators and lattice homomorphisms) which are special to Banach lattices, we are mostly interested here in the Banach space properties of Banach lattices and concentrate on those lattice properties which affect the linear topological or geometry of the underlying Banach space. The basic reference for this aspect of Banach lattices is [15]. For a general introduction to Banach lattices which covers the basic theory as well as some fairly recent material see [1].

The simplest examples of Banach lattices are the spaces with a monotonely unconditional basis under the pointwise order on the coefficients, which as we have seen are better behaved, or at least more regularly behaved, than general Banach spaces. It turns out that much of the structure theory for these spaces carries over to Banach lattices.

It was already mentioned that the L_p and $C(K)$ spaces are Banach lattices. Other examples are the Orlicz spaces and the Lorentz spaces. An *Orlicz function* is an even convex function on $\mathbb{R}$ which is zero at zero and tends to infinity at infinity. If μ is a measure and M is an Orlicz function, the space $L_M(\mu)$ is the collection of all μ-measurable functions f for which there exists $C > 0$

so that $\int M(f/C)\,\mathrm{d}\mu < \infty$. Then $\|f\|_M$ is defined to be the infimum of those $C > 0$ for which $\int M(f/C)\,\mathrm{d}\mu < 1$. This is a norm on $L_M(\mu)$ which makes $L_M(\mu)$ into a Banach lattice. If μ is counting measure on $\mathbb{N}$, $L_M(\mu)$ is called an *Orlicz sequence space* and is denoted by ℓ_M. If $1 \le p < \infty$ and W is a positive nonincreasing continuous function on $(0, \infty)$ so that $W(t) \to \infty$ as $t \to 0+$, $W(t) \to 0$ as $t \to \infty$, $\int_0^1 W(t)\,\mathrm{d}t = 1$, and $\int_0^\infty W(t)\,\mathrm{d}t = \infty$, the *Lorentz space* $L_{W,p}(\mu)$ is the space of all μ measurable functions f for which $\|f\|_{W,p} := \left(\int_0^\infty f^*(t)^p W(t)\,\mathrm{d}t\right)^{1/p}$, where f^* is the decreasing rearrangement of $|f|$. The space $L_{W,p}(\mu)$ is a Banach lattice under the norm $\|f\|_{W,p}$. Lorentz sequence spaces are defined in an analogous fashion.

The Orlicz spaces $L_M(\mu)$ and the Lorentz spaces $L_{W,p}(\mu)$, like the spaces $L_p(\mu)$, are *symmetric* lattices which are *ideals* in the lattice of all μ-measurable functions. Here X is *symmetric* means that if $x \in X$ and y is a μ-measurable function for which $x^* = y^*$, then y is in X and $\|y\|_X = \|x\|_X$. A subspace Y of a lattice X is an *ideal* provided $x \in X$, $y \in Y$, and $|x| \le |y| \Rightarrow x \in Y$. Symmetric lattice ideals play an important role in interpolation theory; see section (11) for a short introduction. (Some sources call a symmetric lattice a *rearrangement invariant space;* other references use "rearrangement invariant space" to mean "symmetric lattice ideal", sometimes with extra conditions. That is why we avoid the term "rearrangement invariant" in this article.)

It is no accident that in the examples of Banach lattices given above the lattice ordering is just the natural pointwise a.e. ordering on a space of (equivalence classes of) real valued functions on a measure space (for $C(K)$ the measure is counting measure on K and for spaces with unconditional basis the measure is counting measure on $\mathbb{N}$). We call such a Banach lattice a *Banach lattice of functions* or a Banach lattice of μ-measurable functions if it is important to specify the measure. There are representation theorems which say that there is essentially no loss of generality in considering only Banach lattices of functions. The most classical of these representation theorems, called the Kakutani representation theorem, gives abstract Banach lattice characterizations of $L_1(\mu)$ and $C(K)$ spaces. A Banach lattice X is an *abstract L_p space,* $1 \le p < \infty$, provided $\|x + y\|^p = \|x\|^p + \|y\|^p$ whenever x and y are *disjoint;* that is, $|x| \wedge |y| = 0$. The L_p version of the Kakutani representation theorem says that **an abstract L_p space is lattice isometric to $L_p(\mu)$ for some measure** μ (see [15, 1.b.2] or [1, Th 12.26]). Moreover, the measure μ can be chosen to be a finite measure if the abstract L_p space X has a *weak order unit;* that is, a vector $u \ge 0$ so that $u \wedge |x| = 0$ only when x is the zero vector. A Banach lattice X is an *abstract M space* provided $\|x + y\| = \|x\| \vee \|y\|$ whenever x and y are disjoint. The $C(K)$ representation theorem says that **an abstract M space is lattice isometric to a sublattice of $C(K)$ for some compact Hausdorff space** K (see [15, 1.b.6]). Moreover, if the abstract M space has a *strong order unit* (that is, a vector $u \ge 0$ such that the unit ball of X is the *order interval* $[-u, u] := \{x : -u \le x \le u\}$), then M is lattice isometric to $C(K)$ itself via an isomorphism which maps u to $\|u\|1_K$.

Suppose that the Banach lattice X admits a *strictly positive* functional; that is, a positive linear functional u^* such that $u^*(|x|) > 0$ for every nonzero vector x in X. Define an (inequivalent) norm $\|\cdot\|_{u^*}$ on X by $\|x\|_{u^*} := u^*(|x|)$. This is obviously an (incomplete) abstract L_1 norm on X from which it follows that the completion X_{u^*} of $(X, \|x\|_{u^*})$ is an abstract L_1 space. Since the formal inclusion operator from X to X_{u^*} is an injective lattice homomorphism, the Kakutani representation theorem yields that X can be thought of as a Banach lattice of functions. If X is separable, so is X_{u^*}, and it follows from comments made in section 4 that X_{u^*} is isometric and order equivalent to a sublattice of $L_1(0,1)$, so that X can be represented as a Banach lattice of Lebesgue measurable functions on the unit interval.

Not every Banach lattice admits a strictly positive functional ($c_0(\Gamma)$ with Γ uncountable is a counterexample), but every separable Banach lattice X does. Indeed, take $\{x_n\}_{n=1}^{\infty}$ dense in the unit sphere of X and for each n pick a norm one functional x_n^* which achieves its norm at x_n. It is easy to check that $u^* := \sum 2^{-n}|x_n^*|$ is a strictly positive functional. From this comment and the representation above it follows that any lattice inequality involving finitely many vectors which is true for lattices of functions must be true in a general Banach lattice. This eliminates the tedium of verifying "obvious" inequalities (for example, $\vee\{\sum_{i=1}^n \epsilon_i x_i : \epsilon_i = \pm 1\} = \sum_{i=1}^n |x_i|$) directly from the axioms for a Banach lattice.

The representation of Banach lattices as lattices of functions suggests that abstract L_1 spaces are particularly important lattices. The abstract M spaces also arise naturally in the general theory. If u is a positive vector in a Banach lattice X and X_u is the linear span of the order interval $[-u, u]$ with $[-u, u]$ taken as the unit ball, then X_u is easily seen to be an abstract M space with strong order unit u (X_u is complete because $[-u, u]$ is closed in the Banach space X).

A Banach lattice is *order complete* or *Dedekind complete* if every nonempty subset which is bounded above has a least upper bound. A dual Banach lattice is order complete; in fact, any *norm* bounded upwarded directed net in a dual Banach lattice converges weak* to the least upper bound of the net.

Suppose that X is order complete and $u \geq 0$ with $\|u\| = 1$. We look a bit more closely at the $C(K)$ space which is lattice isometric to the abstract M space X_u. Given $x \geq 0$, define the *support of* x by $S(x) := \sup_n (nx) \wedge u$. The supremum exists because X is order complete. The support $S(x)$ is a *component* of u. (A vector $0 \leq y \leq u$ is called a component of u [or simply a component if u is understood] provided y is disjoint from $u - y$.) If $T : X_u \to C(K)$ is a lattice isometry with $Tu = 1_K$, then Tx is an indicator function if and only if x is a component. Of course, if $A \subset K$, 1_A is in $C(K)$ if and only if A is clopen (i.e., both open and closed). So the components form a Boolean algebra (a fact which is also easy to verify directly) which is complete because X is order complete. Thus the clopen subsets of K are also a complete Boolean algebra. An important fact is that the clopen subsets of K form a base for the topology of K. One way to see this is to observe that if $0 \le x$ with x in X_u and $t > 0$ satisfies $y := (x - tu) \vee 0 \ne 0$, then $0 \le tS(y) \le x$. The interpretation of this fact in $C(K)$ is that for all $0 \le f$ in $C(K)$ with $f \ne 0$, the set $[f > 0]$ contains a nonempty clopen subset. Since K is compact this implies that the clopen subsets of K form a base for the topology. Using this and the completeness of the Boolean algebra of clopen sets it is a simple exercise to prove that $C(K)$ is itself an order complete Banach lattice. As was pointed out in section 4, this implies that $C(K)$ is 1-injective.

From the discussion in the previous paragraph we can deduce: **If E is a finite dimensional subspace of an order complete lattice X and $\epsilon > 0$, then there is a finite dimensional sublattice F of X and an automorphism T of X so that $E \subset TF$ and $\|I - T\| < \epsilon$.** Given the subspace E, take positive vectors $x_1, \dots, x_n$ in X whose span contains E and normalized so that $u := \max_i x_i$ has norm one. Then $F \subset X_u$ and X_u is isometric to a $C(K)$ space for which the clopen subsets of K form a base for the topology, which means that the span of indicator functions of clopen sets is dense in $C(K)$. Thus fixing a basis $y_1, \dots, y_k$ for E and $\delta > 0$, we get a subspace F of X_u spanned by disjoint vectors and a vector u in F with $\|u\| = 1$ so that for each $1 \le i \le k$, there is a vector x_i so that $|x_i - z_i| \le \delta u$ (and hence $\|x_i - z_i\| \le \delta$). Now apply the principle of small perturbations.

A Banach lattice is *order continuous* if every downward directed net whose greatest lower bound is zero converges in norm (or weakly; it is the same) to zero. This is equivalent to saying that every order bounded increasing sequence converges in norm (necessarily to the least upper bound of the sequence) (see [15, 1.a.8] or [1, Th.12.9] or the beginning of the argument below). It is also easy to check that an order continuous Banach lattice is order complete [15, 1.a.8]. **A Banach lattice is not order continuous if and only if it contains a sequence of disjoint positive vectors which is equivalent to the unit vector basis for c_0 and is bounded above.** The "if" direction is clear. If X is not order continuous, one gets an upward directed net of positive vectors which is bounded above by, say, x, with $\|x\| = 1$. The net cannot converge in norm, so one gets $0 \le x_1 \le x_2 \le \ldots \le x$ so that $a := \inf_n \|x_{n+1} - x_n\| > 0$. Let $y_n := x_{n+1} - x_n \ge 0$. So we have for every n:

$$y_n \ge 0, \quad \|y_n\| \ge a, \quad \sum_{k=1}^{n} y_k \le x. \tag{2}$$

(Although we do not need it here because we want to "disjointify" the y_n's, it is worth noticing that (2) implies that $y_n \to 0$ weakly and hence $\{y_n\}_{n=1}^\infty$ has a basic subsequence, and that any basic subsequence of $\{y_n\}_{n=1}^\infty$ is equivalent to the unit vector basis for c_0.) Take y_n^* in the unit sphere of X^* so that $y_n^*(y_n) = \|y_n\|$. By replacing y_n^* with $|y_n^*|$ if necessary, we may assume that $y_n^* \geq 0$. For each n the nonnegative sum $\sum_m y_n^*(y_m)$ is at most $\|x\|$. It is an elementary exercise in combinatorial reasoning to deduce from this that for any $\epsilon > 0$ there is a subsequence $\{z_k\}_{k=1}^\infty := \{y_{n_k}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ so that for each $k > j$, $z_k^*(z_j) < 2^{-j}\epsilon^2$, where $z_k^* := y_{n_k}^*$. Thus for each n,

$$\| \left(z_{n+1} - \epsilon^{-1} \sum_{k=1}^{n} z_k \right) \vee 0 \| \geq z_{n+1}^* \left(z_{n+1} - \epsilon^{-1} \sum_{k=1}^{n} z_k \right) \geq a - \epsilon.$$

This means that the z_n's have big disjoint pieces. More precisely, let $\epsilon = a/4$ and set $v_{n+1} := (z_{n+1} - \epsilon^{-1} \sum_{k=1}^n z_k) \vee 0$ and $w_n := (v_n - \epsilon x) \vee 0$. Then the w_n's are pairwise disjoint positive vectors smaller than x with norms bounded away from zero.

It is easy to see that $\{w_n\}_{n=1}^\infty$ must be equivalent to the unit vector basis of c_0. This completes the proof, but note that if X is order complete the mapping $e_n \to w_n$ from c_0 into X extends to a mapping from the positive cone of ℓ_∞ to X by defining $\{\alpha_n \delta_n\}_{n=1}^\infty \mapsto \sup_n \alpha_n w_n$; this mapping extends to an order isomorphism from ℓ_∞ into X.

Other useful characterizations of order continuous Banach lattices are given by the following (see [15, 1.b.16] or [1, Th. 12.9]): X **is order continuous if and only if every order interval is weakly compact if and only if** X **is an ideal in** X^{**}.

Since c is not order continuous but is isomorphic to the order continuous Banach lattice c_0, there is not a linear topological characterization of order continuity. While it is only a sufficient condition for order continuity of X that c_0 not embed isomorphically into X, this condition is only a bit too strong as arguments similar to those given above show that **if** c_0 **embeds into** X **then it embeds as a sublattice** (see [1, Th. 14.12]).

The functional representation theorems for order continuous Banach lattices are stronger and more useful than the representation already mentioned for Banach lattices which have a strictly positive functional. Here we just indicate what is going on and refer to [15] for details. However, the motivated reader is encouraged to work out the details for himself or herself (partly because the discussion in [15] wanders unnecessarily). First, it is not hard to show that an order continuous Banach lattice which has a weak order unit u also admits a strictly positive functional u^* (see [15, 1.b.15] or [1, 12.14]) and thus can be thought of as a space of integrable functions on a finite measure space (Ω, μ)

with $u^*(x) = \int x \, d\mu$ for x in X. One can assume that $u^*(u) = 1 = \|u\|$ and $\|u^*\| \leq 2$. Since X is order continuous, one has for each $x \geq 0$ in X that $(nu) \wedge x$ converges to x as $n \to \infty$, so that X_u is dense in X. By replacing the underlying σ-algebra with the smallest σ-algebra for which all the functions in X are measurable, it can be assumed that X is dense in $L_1(\mu)$. Then necessarily $u > 0$ a.e. By replacing the measure μ with $u \, d\mu$ and functions f in X by f/u, one can assume $u \equiv 1$. Now it is possible to recover the μ-measurable sets and the measure μ. From the density of X in $L_1(\mu)$ it follows that the indicator function of a measurable set is an element x in X_u which is a component of u. From this it is essentially obvious that the $L_\infty(\mu)$ norm on X_u agrees with its abstract M space norm. If we represent X_u as a $C(K)$ space, then the sets of positive μ measure are mapped onto the nonempty clopen subsets of K in an obvious way so that we can represent X and $L_1(\mu)$ as function spaces on K and transfer the measure μ to K. We then get that $C(K) \subset X \subset L_1(\mu)$ with both inclusions having dense range. Moreover, $C(K) = L_\infty(\mu)$ and every μ-measurable set is equal μ-a.e. to a clopen subset of K. Finally, X is an ideal in $L_1(\mu)$ (use again order continuity and the fact that X contains all indicator functions).

One consequence of this representation is: **Let X be an order continuous Banach lattice which has a weak order unit. A closed subspace Y of X either embeds into $L_1(\mu)$ for some measure μ or contains a normalized basic sequence which is equivalent to (even equal to a small perturbation of) a disjoint sequence.** For a proof see [15, 1.c.8] or modify the proof of the dichotomy principle discussed in section 4. By applying L_1 theory discussed in section 4 one gets that a subspace of an order continuous Banach lattice is reflexive if and only if it does not contain a subspace isomorphic to either ℓ_1 or c_0 and that a nonreflexive Banach lattice has a sublattice which is order isomorphic to ℓ_1 or c_0. The representation can also be used to prove that a Banach lattice which does not contain a copy of c_0 must be weakly sequentially complete ([15, 1.c.4]).

Although we have used here the representation theorem for abstract L_1 spaces, it should be mentioned that since an abstract L_1 space is necessarily order continuous (every normalized disjoint sequence is obviously 1-equivalent to the unit vector basis of ℓ_1), some of the ideas (particularly building components in X) can be used to prove Kakutani's representation of an abstract L_1 space which has a weak order unit as a space $L_1(\mu)$ for some probability measure μ.

Expressions such as $\left(\sum_{n=1}^N |x_n|^p\right)^{1/p}$, $1 \leq p \leq \infty$, have meaning in a general Banach lattice. (When $p = \infty$ we follow the convention of interpreting $\left(\sum_{n=1}^N |x_n|^p\right)^{1/p}$ to be $\vee_{n=1}^N |x_n|$.) Letting $1/p + 1/p^* = 1$, one can define $\left(\sum_{n=1}^N |x_n|^p\right)^{1/p} = \vee\{\sum_{n=1}^N \alpha_n x_n : \sum_{n=1}^N \alpha_n^{p^*} = 1\}$, but one must justify that this supremum of an infinite set of vectors must exist in the (possibly lattice incomplete) Banach lattice. By taking the supremum over finite sets of scalars satisfying $\sum_{n=1}^N \alpha_n^{p^*} = 1$ one obtains an upward directed net which is bounded above by $\sum_{n=1}^N |x_n|$; it is enough to check that this net converges in X. This is not hard to do directly, but note that if X is a $C(K)$ space the net obviously converges to $\left(\sum_{n=1}^N |x_n|^p\right)^{1/p}$. By working in the abstract M space X_u with $u := \sum_{n=1}^N |x_n|$, the $C(K)$ space case gives the general case. We can then interpret scalar inequalities involving such expressions in general lattices (rather than just in lattices of functions). For example, Khintchine's inequality, discussed in section 4, reads

$$A_p \left(\sum_{n=1}^N |x_n|^2\right)^{1/2} \leq \left(\mathbb{E}|\sum_{n=1}^N \varepsilon_n x_n|^p\right)^{1/p} \leq B_p \left(\sum_{n=1}^N |x_n|^2\right)^{1/2}. \tag{3}$$

Auxiliary lattices such as $X(\ell_p)$ can now be defined and the "obvious" dualities checked. $X(\ell_2^2)$ is especially noteworthy, since it can be regarded as the complexification of X by regarding ℓ_2^2 as $\mathbb{C}$ and defining multiplication by complex scalars in the obvious way (see [15, p. 43] for details).

Starting from the scalar identity $|\alpha|^\theta |b|^{1-\theta} = \inf_{C>0} \theta C^{1/\theta}|\alpha| + (1-\theta)C^{1/(\theta-1)}|\beta|$, one gets in a similar way an interpretation for the expression $|x|^\theta |y|^{1-\theta}$ and that

$$\| |x|^\theta |y|^{1-\theta} \| \leq \|x\|^\theta \|y\|^{1-\theta}. \tag{4}$$

Also, if $1 \leq p < q \leq \infty$, $1/r := \theta/p + (1-\theta)/q$ with $0 < \theta < 1$ and $\alpha_1, \ldots, \alpha_N$ are positive scalars, then for any collection $x_1, \ldots, x_N$ of vectors in a Banach lattice we have that

$$\left(\sum_{n=1}^N \alpha_n |x_n|^r\right)^{1/r} \leq \left(\sum_{n=1}^N \alpha_n |x_n|^p\right)^{\theta/p} \left(\sum_{n=1}^N \alpha_n |x_n|^q\right)^{(1-\theta)/q}. \tag{5}$$

Inequality (5) is obvious for lattices of functions, as is the following: If $1 \leq p \leq \infty$, $x_1, \ldots, x_N$ are vectors in a Banach lattice X and $x_1^*, \ldots, x_N^*$ are in X^*, then

$$\sum_{n=1}^N x_n^*(x_n) \leq \left(\sum_{n=1}^N |x_n^*|^p\right)^{1/p} \left(\sum_{n=1}^N |x_n|^{p^*}\right)^{1/p^*}. \tag{6}$$

A more sophisticated version [15, 1.d.1] of the argument given above shows that for *any* continuous homogeneous ($f(\lambda x) = \lambda f(x)$ if $\lambda \geq 0$) function f on $\mathbb{R}^n$ and $x_1, \ldots, x_n$, the expression $f(x_1, \ldots, x_n)$ can be defined in such a way that any lattice inequality that is true in $\mathbb{R}^n$ is true in X; that is, if g

is another continuous homogeneous function on $\mathbb{R}^n$ and $f(a) \le g(a)$ for all $a = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{R}^n$, then $f(x_1, \ldots, x_n) \le g(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n$ in X.

For $1 \le p \le \infty$, a linear mapping T from a Banach space into a Banach lattice is called *p-convex* if there exists a constant M so that for all finite sets of vectors in the domain space the following inequality holds:

$$\left\| \left(\sum_{n=1}^{N} |Tx_n|^p \right)^{1/p} \right\| \le M \left(\sum_{n=1}^{N} \|x_n\|^p \right)^{1/p}. \tag{7}$$

The smallest such M is denoted by $M^{(p)}(T)$. Clearly $M^{(1)}(T) = \|T\|$. Similarly, if for a linear mapping T from a Banach lattice into a Banach space the inequality

$$\left(\sum_{n=1}^{N} \|Tx_n\|^p \right)^{1/p} \le M \left\| \left(\sum_{n=1}^{N} |x_n|^p \right)^{1/p} \right\| \tag{8}$$

always holds for some constant M, then T is called *p-concave* and the smallest such M is denoted by $M_{(p)}(T)$. Clearly $M_{(\infty)}(T) = \|T\|$. Evidently if T is p-convex [p-concave] then it is bounded and $M^{(p)}(T) \ge \|T\|$ [$M_{(p)}(T) \ge \|T\|$]. It is not hard to check the identities $M^{(p)}(T^*) = M_{(p^*)}(T)$ and $M^{(p)}(T) = M_{(p^*)}(T^*)$, where $1/p + 1/p^* = 1$. As functions of p, $M^{(p)}(T)$ is nondecreasing and $M_{(p)}(T)$ is nonincreasing (see [15, 1.d.5] or write down a proof for $L_p(\mu)$ and see that the Hölder type inequalities (4), (5) allow a translation to the lattice setting).

A Banach lattice X is called *p-convex [p-concave]* if the identity operator I_X on X is p-convex [p-concave] and we then define $M^{(p)}(X) := M^{(p)}(I_X)$ and $M_{(p)}(X) := M_{(p)}(I_X)$. These constants are called the p-convexity and p-concavity constants of X. So X is p-convex [p-concave] if and only if X^* is p^*-concave [p^*-convex].

A p-convex and r-concave Banach lattice can be renormed with an equivalent lattice norm so that the p-convexity and r-concavity constants are both one [15, 1.d.8]. In particular, a lattice which is both p-convex and p-concave is lattice isomorphic to an abstract L_p space.

If $T : X \to Y$ is a positive operator between Banach lattices it is easy to check (see [15, 1.d.9]) the inequality

$$\left\| \left(\sum_{n=1}^{N} |Tx_n|^p \right)^{1/p} \right\| \le \|T\| \left\| \left(\sum_{n=1}^{N} |x_n|^p \right)^{1/p} \right\|. \tag{9}$$

From this it follows that $M^{(p)}(T) \le \|T\| M^{(p)}(X)$ and $M_{(p)}(T) \le \|T\| M_{(p)}(Y)$. There are characterizations [15, 1.d.10] of general operators which are p-convex or p-concave which involve the notion of p-summing operator, to be discussed in section 10. When $p = 2$, the inequality (9) is valid for general operators (except that the right hand side must be multiplied by a constant; see section 10 or [15, 1.f.14]).

The square function $\left(\sum_{n=1}^N |x_n|^2\right)^{1/2}$ of vectors $x_1, \dots, x_N$ in a p-concave, $1 \le p < \infty$, lattice X is very useful for studying unconditional basic sequences and other things as well. The reason is that by taking norms in the lattice Khintchine inequality (3) we get

$$\begin{aligned} A_1 \|(\textstyle\sum_{n=1}^N |x_n|^2)^{1/2}\| \le \|\mathbb{E}\,|\sum_{n=1}^N \varepsilon_n x_n|\| \le \\ \mathbb{E}\,\|\textstyle\sum_{n=1}^N \varepsilon_n x_n\| \le \left(\mathbb{E}\,\|\sum_{n=1}^N \varepsilon_n x_n\|^p\right)^{1/p}. \end{aligned}$$

By p-concavity, this last quantity is dominated by

$$M_{(p)}(X) \| \left(\mathbb{E}\,|\sum_{n=1}^N \varepsilon_n x_n|^p\right)^{1/p} \| \le M_{(p)}(X) B_p \| \left(\sum_{n=1}^N |x_n|^2\right)^{1/2} \|.$$

This gives the equivalence of the norm of the square function of $x_1, \dots, x_n$ with a certain Rademacher average:

$$\begin{aligned} A_1 \| \left(\textstyle\sum_{n=1}^N |x_n|^2\right)^{1/2} \| \le \mathbb{E}\,\|\sum_{n=1}^N \varepsilon_n x_n\| \le \left(\mathbb{E}\,\|\sum_{n=1}^N \varepsilon_n x_n\|^p\right)^{1/p} \\ \le M_{(p)}(X) B_p \| \left(\textstyle\sum_{n=1}^N |x_n|^2\right)^{1/2} \|. \end{aligned} \tag{10}$$

Notice that the monotonicity of $M_{(p)}(X)$ then gives that if X is q-concave for some finite q, then for all finite p there is a constant C_p so that the inequality $\left(\mathbb{E}\,\|\sum_{n=1}^N \varepsilon_n x_n\|^p\right)^{1/p} \le C_p \left(\mathbb{E}\,\|\sum_{n=1}^N \varepsilon_n x_n\|\right)$ holds; this is a special case of the Kahane-Khintchine inequality that will be discussed in section 8.

Inequality (10) implies that if $\{x_n\}_{n=1}^\infty$ is an unconditional basic sequence in a Banach lattice which is p-concave, then the norm $\|\sum_{n=1}^N \alpha_n x_n\|$ of a linear combination is equivalent to the norm of the square function $\left\|\left(\sum_{n=1}^N |\alpha_n x_n|^2\right)^{1/2}\right\|$; more precisely,

$$\begin{aligned} A_1 K^{-1} \left\|\left(\textstyle\sum_{n=1}^N |\alpha_n x_n|^2\right)^{1/2}\right\| \le \|\textstyle\sum_{n=1}^N \alpha_n x_n\| \\ \le M_{(p)}(X) B_p K \left\|\left(\textstyle\sum_{n=1}^N |\alpha_n x_n|^2\right)^{1/2}\right\| \end{aligned} \tag{11}$$

where K is the unconditional constant of $\{x_n\}_{n=1}^{\infty}$. Since the left side of (11) is valid in any lattice, a duality argument yields that if $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for a complemented subspace of *any* Banach lattice, then $\|\sum_{n=1}^{N}\alpha_n x_n\|$ is equivalent to $\left\|\left(\sum_{n=1}^{N}|\alpha_n x_n|^2\right)^{1/2}\right\|$. We write the equivalence in a slightly different way in which it is easier to keep track of the constants. Suppose that $T:Y\to X$, $S:X\to Y$ are operators with $ST=I_Y$ (so that TS is a projection from X onto SY), Y has a monotonely unconditional basis $\{y_n\}_{n=1}^{\infty}$, and X is a Banach lattice. Then:

$$\begin{aligned} A_1\|T\|^{-1}\left\|\left(\sum_{n=1}^{N}|\alpha_n T y_n|^2\right)^{1/2}\right\| &\le \|\sum_{n=1}^{N}\alpha_n y_n\| \\ &\le A_1^{-1}\|S\|\left\|\left(\sum_{n=1}^{N}|\alpha_n T y_n|^2\right)^{1/2}\right\|. \end{aligned} \tag{12}$$

Let $\{y_n^*\}_{n=1}^{\infty}$ be the functionals in Y^* biorthogonal to $\{y_n\}_{n=1}^{\infty}$. By composing S with (contractive) projections onto the span of initial segments of $\{y_n\}_{n=1}^{\infty}$, it can be assumed that Y is finite dimensional, in which case $\{y_n^*\}$ is a monotonely unconditional basis for Y^*. To prove the right inequality in (12), given $\sum_{n=1}^{N}\alpha_n y_n$, choose a norm one functional $\sum_{n=1}^{N}\beta_n y_n^*$ in Y^* so that $\|\sum_{n=1}^{N}\alpha_n y_n\| = \sum_{n=1}^{N}\beta_n y_n^*(\sum_{n=1}^{N}\alpha_n y_n)$. Using (6) and the left inequality in (10), we then get

$$\begin{aligned} \|\sum_{n=1}^{N}\alpha_n y_n\| &= \sum_{n=1}^{N}\beta_n\alpha_n S^* y_n^*(T y_n) \\ &\le \|\left(\sum_{n=1}^{N}|\beta_n S^* y_n^*|^2\right)^{1/2}\|\|\left(\sum_{n=1}^{N}|\alpha_n T y_n|^2\right)^{1/2}\| \\ &\le A_1^{-1}\mathbb{E}\|\sum_{n=1}^{N}\varepsilon_n\beta_n S^* y_n^*\|\|\left(\sum_{n=1}^{N}|\alpha_n T y_n|^2\right)^{1/2}\| \\ &\le A_1^{-1}\|S^*\|\|\sum_{n=1}^{N}\beta_n y_n^*\|\left\|\left(\sum_{n=1}^{N}|\alpha_n T y_n|^2\right)^{1/2}\right\|, \end{aligned}$$

which is the right side of (12).

Continuing in the situation considered in the previous paragraph, we analyze further the case when X is a $C(K)$ space and $\|y_n\|=1$ for all n. Since $\{y_n\}_{n=1}^{\infty}$ is unconditionally monotone, we have $\max_n|\alpha_n|\le\|\sum_{n=1}^{N}\alpha_n y_n\|$. From (12), Hölder's inequality, and properties of the supremum norm we get

$$\begin{aligned} \|\sum_{n=1}^{N}\alpha_n y_n\| &\le A_1^{-1}\|S\|\|(\max_n|\alpha_n T y_n|^{1/2})(\sum_{n=1}^{N}|\alpha_n T y_n|)^{1/2}\| \\ &\le A_1^{-1}\|S\|\left(\max_n|\alpha_n|\|T y_n\|\right)^{1/2}\max_{\pm}\|\sum_{n=1}^{N}\pm\alpha_n T y_n\|^{1/2} \\ &\le A_1^{-1}\|S\|\|T\|\|\sum_{n=1}^{N}\alpha_n y_n\|^{1/2}\max_n|\alpha_n|^{1/2} \end{aligned}$$

so that

$$\| \sum_{n=1}^{N} \alpha_n y_n \| \le \left(A_1^{-1} \|T\| \|S\| \right)^2 \max_n |\alpha_n|.$$

Thus $\{y_n\}_{n=1}^{\infty}$ is K-equivalent ($K := \left(A_1^{-1} \|T\| \|S\| \right)^2$) to the unit vector basis of c_0. Therefore, **a seminormalized unconditional basis for a complemented subspace of a $C(K)$ space is equivalent to the unit vector basis for c_0.** By duality one obtains that **a seminormalized unconditional basis for a complemented subspace of an $L_1(\mu)$ space is equivalent to the unit vector basis for ℓ_1.** Since $C(0,1)$ and $L_1(0,1)$ contain subspaces isomorphic to ℓ_2 while c_0 and ℓ_1 do not, this gives another proof that neither $C(0,1)$ nor $L_1(0,1)$ has an unconditional basis.

By using a p-convexification procedure, it is possible to build a scale of Banach lattices starting with a lattice X in a manner analogous to how the scale of $L_p(\mu)$ spaces is constructed from $L_1(\mu)$. For simplicity, we assume that X is a lattice of μ-measurable functions (for the general case see [15, p. 53]). For $1 < p < \infty$, the *p-convexification* of X is the space $X^{(p)}$ of all μ-measurable functions x for which $|x|^p \text{sign}\,(x)$ is in X. This is easily seen to be a Banach lattice under the norm $\|x\|_{X^{(p)}} := \| \, |x|^p \|_X^{1/p}$. The space $X^{(p)}$ is p-convex with $M^{(p)}(X^{(p)}) = 1$. More generally, if X is r-convex and s-concave, then $X^{(p)}$ is pr-convex and ps-concave with moduli $M^{(pr)}(X^{(p)}) \le M^{(r)}(X)^{1/p}$ and $M_{(ps)}(X^{(p)}) \le M_{(s)}(X)^{1/p}$. If the lattice X is itself p-convex, one can also define the *p-concavification* of X to be the space $X_{(p)}$ of all μ-measurable functions x for which $|x|^{1/p} \text{sign}\,(x)$ is in X. The expression $\| \, |x|^{1/p} \|^p$ is a norm if $M_{(p)}(X) = 1$, which, as we mentioned, can be achieved with an equivalent renorming of X.

6 Geometry of the norm

One interesting and fruitful line of research, dating from the early days of Banach space theory, has been to relate analytic properties of a Banach space to various geometrical conditions on the norm of the Banach space. The simplest example of such a condition is that of *strict convexity*: A Banach space X is *strictly convex* provided that the equality $\|x+y\| = \|x\| + \|y\|$ with y nonzero implies that x is a scalar multiple of y. This just means that the unit sphere of X does not contain any line segment or that every point of the unit sphere is an extreme point of the unit ball. Related to strict convexity is smoothness: A *smooth point* of the unit ball of X is a point x in the unit sphere for which there is only one norm one linear functional which achieves its norm at x. The space X is called *smooth* provided every point in its unit sphere is a smooth point of the unit ball.

The spaces $L_p(\mu)$, $1 < p < \infty$, are strictly convex and smooth, while the spaces $L_1(\mu)$ and $C(K)$ are neither strictly convex nor smooth except in the trivial case when they are one dimensional.

It is easy to check that if X^* is strictly convex [respectively, smooth], then X is smooth [respectively, strictly convex]. The converse is true for reflexive spaces but not in general.

In order to understand the analytic meaning that a point x in the unit sphere of X is a smooth point, consider any other vector y in X and a linear functional x^* on X with $\|x^*\| = x^*(x) = 1$. Then for any $t > 0$, $1 + tx^*(y) = x^*(x + ty) \leq \|x + ty\|$. That is,

$$x^*(y) \leq \frac{\|x + ty\| - \|x\|}{t}.$$

By using the triangle inequality one checks that the function $\frac{\|x+ty\|-\|x\|}{t}$ is an increasing function of t on $(0, \infty)$ and thus

$$x^*(y) \leq \lim_{t \to 0+} \frac{\|x + ty\| - \|x\|}{t}.$$

Similarly,

$$x^*(y) \geq \lim_{t \to 0+} \frac{\|x - ty\| - \|x\|}{-t}.$$

Thus if these two limits coincide, the value of $x^*(y)$ is uniquely determined. On the other hand, if these two limits differ, it follows from the Hahn-Banach theorem that for any λ satisfying

$$\lim_{t \to 0+} \frac{\|x - ty\| - \|x\|}{-t} \leq \lambda \leq \lim_{t \to 0+} \frac{\|x + ty\| - \|x\|}{t}$$

there is a linear functional x^* for which $\|x^*\| = x^*(x) = 1$ and $x^*(y) = \lambda$. Consequently, x is a smooth point if and only if $\|x + ty\| + \|x - ty\| - 2 = o(t)$ for every y in X.

Quantitative versions of strict convexity and smoothness are of particular importance in Banach space theory. A Banach space X is said to be *uniformly convex* provided for every $\epsilon > 0$ there exists $\delta_X(\epsilon) = \delta(\epsilon) > 0$ so that

$$\sup\{\|(x + y)/2\| : \|x\| = \|y\| = 1; \|x - y\| = \epsilon\} = 1 - \delta(\epsilon).$$

The function $\delta_X(\epsilon)$ is called the *modulus of convexity* of X. It is geometrically obvious (and even true) that in the definition of $\delta(\epsilon)$, the equalities can be replaced by the inequalities $\|x\| \le 1$, $\|y\| \le 1$, $\|x-y\| \ge \epsilon$ without changing the value of $\delta(\epsilon)$.

One of the first theorems to relate the geometry of the norm to linear topological properties is the following. **A uniformly convex space is reflexive.** In order to see this, we may assume as well that X is separable. If X is not reflexive, then for any $\lambda > 0$ there is x^{**} in the unit sphere of X^{**} whose distance to X exceeds $1-\lambda$. Let A be a countable subset of the unit sphere of X^* which determines the norm of $Y \equiv \operatorname{span} X \cup \{x^{**}\}$; that is, for each y^{**} in Y, $\|y^{**}\| = \sup\{y^{**}(x^*) : x^* \in A\}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in the unit sphere of X so that $x^*(x_n) \to x^{**}(x^*)$ for every x^* in A. Then $\lim_{n,m\to\infty} \|x_n + x_m\| = 2$, while for every n, $\liminf_{m\to\infty} \|x_n - x_m\| \ge 1-\lambda$. This argument shows that for nonreflexive X, $\delta_X(\epsilon) = 0$ for every $0 < \epsilon < 1$ and in particular that X is not uniformly convex.

A Banach space X is said to be *uniformly smooth* if the function $\rho_X(\tau) = \rho(\tau) = \sup\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}$ satisfies $\rho(\tau) = o(\tau)$ as $\tau \to 0$. Again one checks easily that a uniformly smooth Banach space is reflexive.

There is a complete duality between uniform convexity and uniform smoothness. **The space X is uniformly convex if and only if X^* is uniformly smooth.** This follows from a formula which connects the moduli δ_X and ρ_{X^*}:

$$\rho_{X^*}(\tau) = \sup\{\tau\epsilon/2 - \delta_X(\epsilon) : 0 \le \epsilon \le 2\}.$$

To prove this identity, let x and y be in the unit sphere of X with $\|x-y\| = \epsilon$ and pick x^*, y^* in the unit sphere of X^* so that $x^*(x+y) = \|x+y\|$ and $y^*(x-y) = \|x-y\|$. Then

$$\begin{aligned} 2\rho_{X^*}(\tau) &\ge \|x^* + \tau y^*\| + \|x^* - \tau y^*\| - 2 \\ &\ge (x^* + \tau y^*)(x) + (x^* - \tau y^*)(y) - 2 \\ &= x^*(x+y) + \tau y^*(x-y) - 2 = \|x+y\| + \tau\epsilon - 2. \end{aligned}$$

This shows that $\rho_{X^*}(\tau) \ge \sup\{\tau\epsilon/2 - \delta_X(\epsilon) : 0 \le \epsilon \le 2\}$. Conversely, given $\tau > 0$ and x^*, y^* in the unit sphere of X^*, take x and y in the unit sphere of X so that $(x^* + \tau y^*)(x) = \|x^* + \tau y^*\|$ and $(x^* - \tau y^*)(x) = \|x^* - \tau y^*\|$ (if one wishes to avoid reflexivity at this stage one can choose x and y where the norms of the functionals are almost attained).Then

$$\begin{aligned}\|x^* + \tau y^*\| + \|x^* - \tau y^*\| &\le x^*(x+y) + \tau y^*(x-y) \\ &\le \|x+y\| + \tau\|x-y\| = 2\delta_X(\epsilon) + \epsilon\tau,\end{aligned}$$

where $\epsilon = \|x-y\|$. This proves the reverse inequality.

The moduli of convexity and smoothness of a Hilbert space H can be easily computed from the parallelogram identity:

$$\delta_H(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}} = \frac{\epsilon^2}{8} + O(\epsilon^4)$$

$$\rho_H(\tau) = \sqrt{1+\tau^2} - 1 = \frac{\tau^2}{2} + O(\tau^4)$$

Hilbert spaces are the "most uniformly convex" and "most uniformly smooth" spaces in the sense that if X is any space whose dimension is at least two, then $\delta_H(\epsilon) \ge \delta_X(\epsilon)$ and $\rho_H(\tau) \le \rho_X(\tau)$. For infinite dimensional X this is an immediate consequence of Dvoretzky's theorem, to be discussed in section 8, but a direct 2-dimensional geometrical argument gives the general case.

The moduli of convexity and smoothness of the L_p spaces can be computed exactly. It is somewhat easier to describe their asymptotic behavior near zero, and this is what really matters for Banach space theory. The result is

$$\delta_{L_p}(\epsilon) = \begin{cases} (p-1)\frac{\epsilon^2}{8} + o(\epsilon^2), & \text{if } 1 < p \le 2 \\ \frac{\epsilon^p}{p2^p}, & \text{if } 2 \le p < \infty. \end{cases}$$

$$\rho_{L_p}(\tau) = \begin{cases} \frac{\tau^p}{p} + o(\tau^p), & \text{if } 1 < p \le 2 \\ (p-1)\frac{\tau^2}{2} + o(\tau^2), & \text{if } 2 \le p < \infty. \end{cases}$$

This result can be extended to Banach lattices. If X is a Banach lattice which is p-convex with $1 < p \le 2$ and q-concave with $2 \le q < \infty$ and $M^{(p)}(X) = 1 = M_{(q)}(X)$, then $\delta_X(\epsilon) \ge \epsilon^q/C$ and $\rho_X(\tau) \le C\tau^p$ for some constant $C = C(p,q) > 0$ (see [15, 1f.1]).

Since it is easier to do analysis on a Banach space which has a norm with good geometric properties than on a general space, it is important to know which Banach spaces can be equivalently renormed so as to become strictly convex or smooth or uniformly convex or uniformly smooth, and it is useful to have in these last cases moduli which are as good as possible. The simplest, but nevertheless useful, renorming technique is as follows. Suppose that T is an injective operator from a Banach space X into a strictly convex Banach space Y. Then it is easy to check that $|||x||| := \|x\| + \|Tx\|$ is an equivalent strictly convex norm on X. Moreover, if T is an isomorphism and Y is uniformly convex, then $|||\cdot|||$ is an equivalent uniformly convex norm on X. Now if X is separable, then there is an injective operator T from X into ℓ_2, so $\|x\|_1 = \|x\| + \|Tx\|$ is an equivalent strictly convex norm on X. Also, there is an operator S from ℓ_2 into X with dense range, so S^* is an injective operator from X^* into $\ell_2^* = \ell_2$ and $\|x^*\|_2 = \|x^*\| + \|S^*x^*\|$ defines an equivalent strictly convex norm on X^*. Since the adjoint operator S^* is weak* to weak continuous, $\|\cdot\|_2$ is dual to a (necessarily smooth) norm $\|\cdot\|_2$ on X. If X is smooth and T is an injective operator from X into ℓ_2 then $(\|x\|^2 + \|Tx\|_2^2)^{1/2}$ is an equivalent norm which is both strictly convex and smooth. Hence **every separable Banach space has an equivalent norm which is both strictly convex and smooth.**

For certain nonseparable spaces; in particular, $\ell_\infty(\Gamma)$ with Γ uncountable (see [6, Ch. II.7]), there may be no equivalent strictly convex or smooth norm.

If we are interested in obtaining a uniformly convex or smooth equivalent norm we have to restrict attention to reflexive spaces. However, not every reflexive space can be so renormed. For example, if $(\sum_{n=1}^{\infty} \ell_1^n)_2$ had an equivalent uniformly convex norm $\|\cdot\|$, and $\|\cdot\|_n$ denotes the restriction of $\|\cdot\|$to the nth coordinate space ℓ_1^n, then the expression $|||x||| := \lim \|x\|_n$ (where "lim" is interpreted to be a limit over some free ultrafilter on $\mathbb{N}$ or a Banach limit or the limit along an appropriate subsequence) defines an equivalent norm on the finitely supported vectors in ℓ_1 which extends uniquely to an equivalent uniformly convex norm on ℓ_1, but this is impossible.

There is a characterization of those spaces on which there is an equivalent uniformly convex norm. These spaces, called *superreflexive spaces*, are discussed in section 9. The superreflexive spaces are also the class of spaces on which there is an equivalent uniformly smooth norm. (These deep facts are discussed in [29].) If a space X has an equivalent uniformly convex norm $|||\cdot|||$, then the equivalent uniformly convex norms are dense in the metric space of equivalent norms (considered as bounded functions on the unit sphere of X). Take, for example, $\|\cdot\| + \epsilon\, |||\cdot|||$. The uniformly convex equivalent norms also form a G_δ set since those whose modulus of convexity at $1/n$ is positive forms an open set. Thus the equivalent uniformly convex norms on X is a dense G_δ in the space of equivalent norms on X. Since, as we mentioned, X^* also admits an equivalent uniformly convex norm, it follows by duality that the equivalent uniformly smooth norms on X is also a dense G_δ in the space of equivalent norms on X, hence so is the family of equivalent norms which are simultaneously uniformly convex and uniformly smooth.

In section 8 it is pointed out that an infinite dimensional $L_p(\mu)$ space cannot be equivalently renormed so as to have a better modulus of convexity or smoothness than that of the natural norm.

There has long been a desire to describe reflexivity geometrically (that is, to show that a space is reflexive if and only if there is an equivalent norm on the space that satisfies some geometrical condition), and the notion of uniform convexity pushed in that direction. This problem was recently solved (at least for separable spaces) and is discussed in [29].

Early in a first course in functional analysis a student learns that a Banach space is reflexive if and only if its closed unit ball is weakly compact. There is a beautiful and useful characterization of weak compactness, called James' theorem, which does not explicitly involve the weak topology: **A nonempty closed convex subset C of a Banach space X is weakly compact if and only if every x^* in X^* attains its maximum on C.** The only if part is of course trivial. For the hard direction see [11, Th. 79] or [26] for an accessible proof when X is separable. This theorem says that on a nonreflexive space X there exist linear functionals which do not attain their norm on the unit ball of X. Nevertheless, the functionals which attain their norm on the unit ball is a rich set: The Bishop-Phelps theorem says: **Let C be a nonempty closed bounded subset of a Banach space X. Then the functionals which attain their maximum on C is (norm) dense in X^*.** This theorem, which is the starting point of the theory of optimization on Banach spaces, has many extensions and applications (see [25]).

We outline the proof of the Bishop-Phelps theorem. First note that if f is a continuous bounded function on a complete metric space (U, d), then there is, for every $\epsilon > 0$, a point u_0 in U so that $f(u) \le f(u_0) + \epsilon d(u, u_0)$ for every u in U. Indeed, define a partial order on U by $u << v$ if $f(u) \le f(v) - \epsilon d(u, v)$. The boundedness of f and completeness of U imply via Zornication that there is a maximal element u_0 in this partial order. This u_0 evidently has the desired property.

Suppose now that C is a nonempty closed bounded convex subset of X, $\epsilon > 0$, and x^* is in X^*. By the remark above there is a point x_0 in C so that $x^*(x_0) \ge x^*(x) - \epsilon\|x - x_0\|$ for all x in C. Consider two convex sets K_1 and K_2 in $X \oplus \mathbb{R}$:

$$K_1 := \{(x, t) : x \in C;\ x^*(x) \ge t\}$$
$$K_2 := \{(x, t) : x \in X;\ t \ge x^*(x_0) + \epsilon\|x - x_0\|\}.$$

The set K_2 has a nonempty interior which is disjoint from K_1, so the separation theorem gives a nonzero point (u^*, α) in $X^* \oplus \mathbb{R}$ and a β so that $u^*(x) + \alpha t \ge \beta$ for (x, t) in K_1 and $u^*(x) + \alpha t \le \beta$ for (x, t) in K_2. Clearly α must be negative, so by normalizing we may assume that $\alpha = -1$. Since $(x_0, x^*(x_0))$ is in $K_1 \cap K_2$, it follows that $\beta = u^*(x_0) - x^*(x_0)$. By considering K_1 we see that $x^* - u^*$ attains its maximum on C at x_0 and by considering K_2 we deduce that $\|u^*\| \leq \epsilon$.

The proof of the Bishop-Phelps theorem uses heavily the order in the real field. In fact, the Bishop-Phelps theorem does not hold in the complex case. There is a nonempty bounded closed convex set C in a complex Banach space X so that for every x^* in X^*, $|x^*|$ does not attain its maximum on C. This is discussed in [26].

We conclude this section by mentioning the geometric meaning of the Radon-Nikodým property or RNP, an analytical concept that will be discussed in section 7. A *slice* of a closed bounded convex set C is a set of the form $S(C, x^*, \alpha) = \{x \in C : x^*(x) \geq \sup_{y \in C} x^*(y) - \alpha\}$ with x^* in X^* and $\alpha > 0$. C is called *dentable* provided that for each $\epsilon > 0$ there is a slice of C which has diameter smaller than ϵ. In [8, Th. V.3.9] and [3, Th. 5.8] it is shown that X **has the RNP if and only if every nonempty closed bounded convex subset of X is dentable.** Here we show how dentability can be used to derive other geometric conditions, in particular, a version of the Krein-Milman theorem valid for noncompact subsets of a space which has the RNP. **If X has the RNP, then every closed bounded convex subset C of X is the closed convex hull of its extreme points.** Recall that a *face* of a convex set C is a nonempty (necessarily convex) subset F such that if $\lambda x + (1-\lambda)y \in F$ with x, y in C and $0 < \lambda < 1$ then x and y are in F. Extreme points are faces consisting of a single point. From the definition of slice it is obvious that if $|(x^* - y^*)(y)| \leq \delta$ for all y in C then $S(C, y^*, \alpha - 2\delta) \subset S(C, x^*, \alpha)$ as long as $\alpha > 2\delta$. We first show that C has an extreme point. Take any slice $S(C, x^*, \alpha)$ of C whose diameter is less than one. By the observation above and the Bishop-Phelps theorem, there is a y^* arbitrarily near x^* which attains a maximum on C and so that the points P_1 in C at which y^* attains its maximum is contained in the slice $S(C, x^*, \alpha)$ and thus has diameter less than one. Evidently P_1 is a closed face of C and of course is dentable since X has the RNP. By induction we get a sequence $\{P_n\}_{n=1}^{\infty}$ so that P_{n+1} is a face of P_n and $\operatorname{diam} P_n \leq 1/n$. Since a face of a face is a face, all the P_n's are faces of C and their intersection is an extreme point of C which is in the slice $S(C, x^*, \alpha)$. To finish the proof, let K be the closed convex hull of the extreme points of C. If K were properly contained in C, then the separation theorem and the Bishop-Phelps theorem would yield a functional x^* which attains a maximum on C and so that the set of points P in C at which x^* attains its maximum is disjoint from K. By what we have proved P has an extreme point which is a fortiori an extreme point of C, which contradicts the definition of K.

It is open whether this extreme point property for every nonempty closed bounded convex subset of a space actually characterizes spaces with the RNP.

The statement about extreme points is however valid in a stronger form which trivially implies that the dentability condition is equivalent to the RNP. A point x in C is called a *exposed point* of C if there is x^* in X^* which attains its maximum on C exactly at the single point x. If also the diameter of the slice $S(C, x^*, \alpha)$ tends to zero as $\alpha \to 0+$, then x is called a *strongly exposed point* of C. For example, every point on the unit sphere of a uniformly convex space is a strongly exposed point of the unit ball. The stronger statement to which we alluded above is: **Every nonempty closed bounded convex subset of a space with the RNP is the closed convex hull of its strongly exposed points.**

7 Analysis in Banach spaces

In this section we describe the basic facts concerning integration and differentiation in Banach spaces as well as the connection of these topics to convexity. Our "point of view" is to take as known the scalar theory but not assume that the reader has any familiarity with the vector valued theory. Much of what we treat in this section is contained either in [8] or in [11], [12]. Almost everything is in [3].

Let (Ω, μ) be a complete σ-finite measure space and X a Banach space. A function from $\Omega \to X$ of the form $\sum_{i=1}^n x_i 1_{A_i}$ with each A_i a measurable subset of Ω is called a *simple function.* A function $f : \Omega \to X$ is called *strongly measurable* or just *measurable* if it is the limit almost everywhere of a sequence of simple functions. A function f is called *scalarly measurable* provided $x^* f$ is measurable for each x^* in X^*. A function f is measurable if and only if it is scalarly measurable and the range of f is essentially separable, which means that for some set A of measure zero, $f[\Omega \sim A]$ is separable. The only if assertion is clear. To verify the if part, we may assume that X is separable and select a sequence $\{x_n^*\}_{n=1}^\infty$ in the unit ball of X^* so that $\|x\| = \sup_n x_n^*(x)$ for every x in X. Thus for each fixed x in X, the function $\|f - x\|$ is a measurable real valued function on Ω. Since X is separable, given $\epsilon > 0$ there is a sequence $\{x_n\}_{n=1}^\infty$ in X and a sequence $\{A_n\}_{n=1}^\infty$ of disjoint measurable subsets of Ω so that $\|f - \sum_n x_n 1_{A_n}\| < \epsilon$. That is, f can be approximated uniformly by functions taking only countably many values and having all level sets measurable and hence f is the limit a.e. of a sequence of simple functions.

We next define the notion of the Bochner integral of a vector valued function. This is the "strongest" of the various vector valued integrals which have been considered and is the one most useful for the topics treated in this Handbook.

If f is a simple function supported on a set of finite measure, define $\int f \, \mathrm{d}\mu = \sum_{x \in X} \mu[f = x]x$. This is of course a finite sum. It is easy to see directly, and also follows from the scalar case by composing with linear functionals, that the integral is linear on this class of simple functions. If f is the a.e. limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions supported on sets of finite measure and $\lim_{n,m} \int \|f_n - f_m\| \, \mathrm{d}\mu = 0$, then $\int f \, \mathrm{d}\mu$ is defined to be $\lim_n \int f_n \, \mathrm{d}\mu$. It is easy to check that $\int f \, \mathrm{d}\mu$ does not depend on the particular sequence $\{f_n\}_{n=1}^{\infty}$ and that $\int f \, \mathrm{d}\mu$ exists if and only if f is measurable and $\int \|f\| \, \mathrm{d}\mu < \infty$. That the integral has the expected properties either follows from the scalar case by composing with linear functionals or is easy to check directly; see [8, pp. 44-52]. In particular, $\| \int f \, \mathrm{d}\mu\| \le \int \|f\| \, \mathrm{d}\mu$ and $T \int f \, \mathrm{d}\mu = \int Tf \, \mathrm{d}\mu$ whenever f is integrable and T is an operator.

That the usual differentiability properties hold for the Bochner integral can be deduced from the scalar theorems even without recalling the proofs in that case. Suppose that f is a Lebesgue integrable X valued function on $\mathbb{R}^n$. Then for a.e. u in $\mathbb{R}^n$, we have

$$\lim_{r\to 0} m(B(0,r))^{-1} \int\limits_{B(u,r)} \|f(v) - f(u)\| \, \mathrm{d}v = 0 \tag{13}$$

and thus also $f(u) = \lim_{r\to 0} m(B(0,r))^{-1} \int_{B(u,r)} f(v) \, \mathrm{d}v$ (here integration is with respect to Lebesgue measure m on $\mathbb{R}^n$). Indeed, we may assume that X is separable and $\{x_n\}_{n=1}^{\infty}$ is dense in X. Then by the scalar theorem,

$$\lim_{r\to 0} m(B(0,r))^{-1} \int\limits_{B(u,r)} \|f(v) - x_n\| \, \mathrm{d}v = \|f(u) - x_n\|$$

for a.e. u and every n. For a point u for which this holds for every n we have

$$\begin{aligned} &\limsup\nolimits_{r\to 0} m(B(0,r))^{-1} \textstyle\int_{B(u,r)} \|f(v) - f(u)\| \, \mathrm{d}v \le \\ &\limsup\nolimits_{r\to 0} m(B(0,r))^{-1} \textstyle\int_{B(u,r)} \|f(v) - x_n\| + \|x_n - f(u)\| \, \mathrm{d}v \\ &\qquad = 2\|f(u) - x_n\|. \end{aligned}$$

Since $\{x_n\}_{n=1}^{\infty}$ is dense in X we get (13), as desired. In particular, if $f : \mathbb{R} \to X$ is integrable and $F(t) = \int_0^t f$ then $F'(s) = f(s)$ a.e.

The Banach space $L_p(\mu, X)$, $1 \le p < \infty$, is defined to be the space of all measurable X valued functions for which $\|f\|_p := (\int \|f\|^p \, \mathrm{d}\mu)^{1/p} < \infty$ (with the usual modification when $p = \infty$). The simple functions which are supported on sets of finite measure are dense in $L_p(\mu, X)$, $1 \le p < \infty$. When μ is counting measure, $L_p(\mu, X) = \ell_p(X) = (\sum X)_p$.

We mention two important notions of differentiability for a function f from a

Banach space X into a Banach space Y. f is said to be *Gâteaux differentiable* or *G-differentiable* at a point x_0 in X if there is an operator T from X into Y so that for every u in X $\lim_{t\to 0}(f(x_0+tu)-f(x_0))/t = Tu$. If this limit exists uniformly with respect to u in the unit sphere of X the function f is said to be *Fréchet differentiable* or *F-differentiable* at x_0. That is, f is F-differentiable at x_0 provided $f(x_0+u) = f(x_0) + Tu + o(\|u\|)$. The operator T is unique if it exists and is called the G or F derivative of f at x_0 and is denoted by $D_f(x_0)$. If f is F-differentiable at x_0 it is continuous there and standard two dimensional examples show that G-differentiability does not imply continuity. However, if f is Lipschitz and X is finite dimensional, then G-differentiability is easily seen to imply F-differentiability. This is false for infinite dimensional spaces and this is one of the main causes for the difficulty of the subject of derivatives in infinite dimensions. For example, the function $f(u) = \sin u$ is an everywhere G-differentiable Lipschitz map from $X = L_2(0,1)$ into itself with $D_f(u)(v) = \cos u \cdot v$, but f is nowhere F-differentiable (for example, at $u_0 = 0$, $\sin 1_{(0,t)} - 1_{(0,t)} = (\sin 1 - 1)1_{(0,t)}$ is not $o(\|1_{(0,t)}\|_2)$ as $t \to 0$).

In general, familiar finite dimensional theorems (chain rule, inverse function theorem, implicit function theorem, etc.) go over to F-derivatives with the same proofs, while for G-derivatives one must proceed with care. On the other hand, the existence theorems for G-derivatives are quite satisfactory while F-derivatives often do not exist and the question when they exist leads to difficult problems. See [39] for a discussion of this topic.

A direct application of the scalar mean value theorem yields that if the G-derivative $D_f(x)$ exists in a neighborhood of x_0 and is continuous at x_0, then it is actually an F-derivative. Thus the notion of C^1 function is the same in the context of G or F derivatives. For C^1 functions we can speak of the second derivative in either the G or F sense. The second derivative is then an operator from X into the space of operators from X to Y or, alternatively, a bounded bilinear map from $X \oplus X$ to Y. Similarly, one can speak of C^n or C^∞ functions. One can define analytic functions when X and Y are complex spaces as those C^∞ functions f for which the Taylor expansion converges in some neighborhood and represents f there. A useful fact that is proved in elementary textbooks (such as [19, Ch. 3]) is that a function f from the complex plane into Y is analytic if y^*f is analytic for each y^* in Y^*.

One important Banach space theory property that comes from differentiation theory is the *Radon-Nikodým property* or RNP. One way to define this notion is the following: The space X *has the RNP if every Lipschitz function from* $\mathbb{R}$ *into* X *is differentiable a.e.* Here we do not need to specify Gâteaux or Fréchet differentiability since the notions obviously coincide for all functions from $\mathbb{R}$. The RNP is usually defined by the requirement that the Radon-Nikodým theorem holds for X valued measures of finite total variation (see for example [8]). We shall discuss that aspect and other equivalences of the RNP later in this section. Here just note that a Banach space valued Lipschitz function f on $[0, 1]$ has separable range, so its a.e. derivative f', if it exists, is also separably valued and thus is a measurable function since $x^* f$ is measurable for each x^* in X^*.

First let us get a feeling for which spaces have the RNP. It is clear from the definition that a subspace of a space with the RNP has the RNP and that a space all of whose separable subspaces have the RNP has the RNP as well. The mapping $t \mapsto 1_{(0,t)}$ from $(0, 1)$ into $L_1(0, 1)$ shows that $L_1(0, 1)$ fails the RNP. Also c_0 fails the RNP. This is seen by considering the function $f : \mathbb{R} \to c_0 \oplus_\infty c_0$ defined by $f(t) = \{\int_0^t \sin ns \, \mathrm{d}s\}_{n=1}^\infty \oplus \{\int_0^t \cos ns \, \mathrm{d}s\}_{n=1}^\infty$. That f maps into c_0 follows from the Riemann-Lebesgue lemma but the only possible candidate for a derivative is $\{\sin ns\}_{n=1}^\infty \oplus \{\cos ns\}_{n=1}^\infty$ which is not in c_0 for any s.

Separable conjugate spaces have the RNP and thus all reflexive spaces have the RNP (incidentally, this yields another proof of the fact mentioned in section 3 that $L_1(0, 1)$ does not embed into a separable conjugate space). To see this, let f be a Lipschitz function into a separable conjugate space Z^* and let $\{z_n\}_{n=1}^\infty$ be dense in Z. For all n, the scalar Lipschitz function $f(t)(z_n)$ is differentiable for a.e. t. At a point t_0 where all of these functions are differentiable, $f(t_0)(z)$ is differentiable for every z in Z (observe that $h^{-1} (f(t_0 + h) - f(t_0)) (z) - k^{-1} (f(t_0 + k) - f(t_0)) (z) \to 0$ as $h, k \to 0$ because the difference quotient is uniformly bounded since f is Lipschitz and tends to zero on the dense set $\{z_n\}_{n=1}^\infty$). From this we conclude that the limit $g(t) := \lim_{h\to 0} h^{-1} (f(t + h) - f(t))$ exists a.e., but the limit is only in the weak* sense. This is all that can be said using just the separability of Z. However, since Z^* is separable, we deduce that g is measurable (g has separable range and $\|g(t) - z^*\|$ is clearly measurable for every z^* in Z^*; this is all that was used in the proof of measurability in the beginning of this section). Also g is bounded by the Lipschitz constant of f, so the Bochner integral $G(s) := \int_0^s g(t) \, \mathrm{d}t$ is well defined for all s. Evaluating both sides at an arbitrary z in Z we see from the scalar theory that $G(s)(z) = (f(s) - f(0)) (z)$ so that $f(s) = G(s) + f(0)$ and hence by what we proved on the differentiation of the integral we conclude that $f'(s) = G'(s) = g(s)$ a.e. also in the sense that $\|h^{-1} (f(s + h) - f(s)) - g(s)\| \to 0$ as $h \to 0$.

Thus every subspace of a separable conjugate space has the RNP and a space with the RNP has no subspace isomorphic to c_0 or $L_1(0, 1)$. These facts give useful criteria for the RNP and in certain cases, such as spaces with an unconditional basis, allow a complete determination whether a space has the RNP. However, there do exist separable spaces with the RNP which do not embed into a separable conjugate space and there are spaces failing the RNP which do not contain isomorphic copies of either c_0 or $L_1(0, 1)$ (see [3, section 3.4]).

Before stating the usual definition of the RNP we need to introduce the notion of what is termed in [8] a "countably additive vector measure on a σ-algebra". If X is a Banach space, *an X valued measure τ is a countably additive function from a σ-algebra into X.* For this to make sense, the series $\sum \tau(A_n)$ must converge unconditionally for each sequence $\{A_n\}_{n=1}^{\infty}$ of disjoint sets in the σ-algebra. One defines the *total variation* $|\tau|$ of an X valued measure by $|\tau|(A) = \sup \sum_n \|\tau(A_n)\|$, where the supremum is over all partitions of A into finitely many (or countably many; it is the same) disjoint sets in the σ-algebra $(\Omega, \mathcal{B})$. The measure τ is said to be of *finite variation* provided $|\tau|(\Omega) < \infty$. If τ is of finite variation, then $|\tau|$ is a finite scalar measure. For example, if $\Omega = \mathbb{N}$, $\mathcal{B}$ is the collection of all subsets of $\mathbb{N}$, and $\{x_n\}_{n=1}^{\infty}$ is a sequence in the Banach space X, then the assignment $\tau\{n\} := x_n$ extends to an X valued measure on $\mathcal{B}$ if and only if $\sum_n x_n$ converges unconditionally. The measure is then of finite variation if and only if $\sum_n \|x_n\| < \infty$. An example of an $L_p(0,1)$, $1 \le p < \infty$, valued measure is gotten by taking the Lebesgue measurable subsets of $[0,1]$ and defining $\tau(A) = 1_A$. This measure has finite variation only if $p = 1$.

The theory of X valued measures and more general vector measures is exposed in [8]. In addition to their importance for the geometry of Banach spaces, vector measures are centrally important for spectral theory; the ones encountered there typically do not have finite variation.

An X valued measure τ is *absolutely continuous* with respect to the *finite* scalar measure μ provided $\tau(A) = 0$ for every set of μ measure zero. Because μ is finite, this is equivalent to saying that for every $\epsilon > 0$ there is $\delta > 0$ so that $\|\tau(A)\| < \epsilon$ whenever $\mu(A) < \delta$ (see [8, p. 10]). If τ is an X valued measure of finite variation it is clear that τ is absolutely continuous with respect to its total variation $|\tau|$ and even satisfies (14) below (with $\mu := |\tau|$).

An X valued measure τ is *differentiable with respect to a scalar measure μ* provided that there is an X valued measurable function g so that $\tau(A) = \int_A g \, \mathrm{d}\mu$ for every measurable set A. We say that the *Radon-Nikodým theorem holds in X* provided that if τ is an X valued measure of finite variation and τ is absolutely continuous with respect to a finite scalar measure μ, then τ is differentiable with respect to μ. If X satisfies this condition only for all separable finite scalar measures, we say that *the separable Radon-Nikodým theorem holds in X* (a measure μ is called *separable* provided $L_1(\mu)$ is separable). The usual definition is that *a Banach space X has the RNP provided the Radon-Nikodým theorem holds in X* and this is equivalent to saying that the separable Radon-Nikodým theorem holds in X (see [8, Ch. III]). Later we prove this equivalence for separable X, but first we show **a general space X has the RNP if and only if the separable Radon-Nikodým theorem holds in X.**

Suppose that the separable Radon-Nikodým theorem holds in X and let $f : [0,1] \to X$ be a Lipschitz function. One defines a linear mapping T from the step functions on $[0,1]$ into X by setting $T1_{[a,b]} := f(b) - f(a)$ for a subinterval of $[0,1]$ and extending linearly. Since f is a Lipschitz function, the mapping T is continuous when the step functions are given the $L_1(0,1)$ norm, and hence T uniquely extends to an operator (also denoted by T) from $L_1(0,1)$ into X. The assignment $\tau(A) := T1_A$ obviously defines an X valued measure of finite variation which is absolutely continuous with respect to Lebesgue measure m, so we get an X valued measurable function g on $[0,1]$ for which $\tau(A) = \int_A g \, \mathrm{d}m$ for every Lebesgue measurable subset of $[0,1]$. In particular, $f(t) = \int_0^t g \, \mathrm{d}m + f(0)$ and thus by what we proved in the beginning of this section $f'(t)$ exists a.e. on $[0,1]$ (and is equal to $g(t)$).

Suppose that X has the RNP. To see that the separable Radon-Nikodým theorem holds in X, suppose first that the scalar "control measure" μ is Lebesgue measure on $[0,1]$ and that the X valued measure τ satisfies

$$\|\tau(A)\| \leq \mu(A) \tag{14}$$

for all Lebesgue measurable subsets of $[0,1]$. Define $f(t) = \tau[0,t]$. Evidently f is Lipschitz and so by assumption is differentiable a.e. on $[0,1]$ and f' is measurable. That $\tau(A) = \int_A f' \, \mathrm{d}\mu$ for all Lebesgue measurable sets now follows from the scalar theory by composing with linear functionals.

By the measure isomorphism theorem already used in section 4, we get that if the finite control measure μ is separable and the X valued measure τ satisfies (14), then τ is differentiable with respect to μ.

Now suppose that ν is a finite separable measure and τ is an X valued measure which is absolutely continuous with respect to ν. Then $\mu := |\tau|$ is a finite scalar measure which is absolutely continuous with respect to ν, so by the scalar Radon-Nikodým theorem there is a ν-measurable function $f \geq 0$ so that $\mu(A) = \int_A f \, \mathrm{d}\nu$ for every ν-measurable set A. Of course, μ is then also a separable measure and, as we have already remarked, τ satisfies (14), so from what we already have proved there is an X valued μ-measurable function g so that $\tau(A) = \int_A g \, \mathrm{d}\mu$ for every μ-measurable set A. Then $f \cdot g$ is ν-measurable and $\tau(A) = \int_A f \cdot g \, \mathrm{d}\nu$ for every ν-measurable set A.

Observe that the simple argument reducing the study of an X valued measure which is absolutely continuous with respect to a finite control measure to the case where the X valued measure satisfies (14) yields another characterization of the RNP; more precisely, that **the [separable] Radon-Nikodým theorem holds in X if and only if for each operator T from an $L_1(\mu)$ space with μ finite [and separable] into X there is an X valued measurable function g so that $Tf = \int fg \, \mathrm{d}\mu$ for all f in $L_1(\mu)$.** From this it is easy to see that if X is separable and the separable Radon-Nikodým theorem holds in X then the Radon-Nikodým theorem holds in X. Indeed, let μ be a finite measure on a σ-algebra $\mathcal{B}$ and let $T : L_1(\mu) \to X$ be an operator. Since X is separable there is a sequence $\{x_n^*\}_{n=1}^{\infty}$ in X^* which separates the points of X. Let $\mathcal{A}$ be a countably generated sub σ-algebra of $\mathcal{B}$ so that all the $L_\infty(\mu)$ functions $T^*x_n^*$ are $\mathcal{A}$-measurable. Since $\{x_n^*\}_{n=1}^{\infty}$ separates points of X, $Tf = T\mathbb{E}(f|\mathcal{A})$ for each f in $L_1(\mu)$. The restriction of μ to $\mathcal{A}$ is a separable measure since $\mathcal{A}$ is countably generated. Thus we get an X valued $\mathcal{A}$-measurable function g so that $Tf = \int f \cdot g \, \mathrm{d}\mu$ for each $\mathcal{A}$-measurable function f in $L_1(\mu)$. But then for a general f in $L_1(\mu)$ we have $Tf = T\mathbb{E}(f|\mathcal{A}) = \int \mathbb{E}(f|\mathcal{A})g \, \mathrm{d}\mu = \int f \cdot g \, \mathrm{d}\mu$ since g is $\mathcal{A}$-measurable.

One of many places where the RNP arises naturally is in the study of vector valued L_p spaces. There is a natural isometric identification of $L_{p^*}(\mu, X^*)$, $1/p + 1/p^* = 1$, with a subspace of $L_p(\mu, X)^*$, and for $1 \le p < \infty$, $L_p(\mu, X)^* = L_{p^*}(\mu, X^*)$ **for all finite (or σ-finite) measures μ if and only if X^* has the RNP** (see [8, Ch. IV]).

There are other important analytic characterizations of spaces with the RNP in terms of martingales. In particular, the RNP spaces are exactly those Banach spaces in which the martingale convergence theorem is valid in the sense that X **has the RNP if and only if every L_1 bounded X valued martingale converges a.e.** (see [8, Ch. V]).

It turns out that in many places where one might assume reflexivity in order to use weak compactness of the unit ball it suffices to assume that the space has the RNP.

We now discuss the differentiability of (real valued) convex continuous functions on a Banach space X. Part of the importance of this topic derives from the fact that the norm is a convex continuous function and differentiability of the norm is intrinsically related to its smoothness. The most elementary reference for the differentiability of convex functions and related topics is probably [11, Ch. 5].

An easy consequence of the definition is that a locally bounded convex function is continuous and even locally Lipschitz. By using the separation theorem in $X \oplus \mathbb{R}$ it follows that whenever f is convex and continuous in a neighborhood of a point x_0 the set (called the *subdifferential of f*) $\partial_f(x_0) := \{x^* \in X^* : x^*(x - x_0) \le f(x) - f(x_0) \text{ for all } x \in X\}$ is nonempty. From the theory of convex functions on $\mathbb{R}$ we know that for each u the right and left derivatives of the function $t \mapsto f(x_0 + tu)$ exist at $t = 0$. These one-sided derivatives agree for every u (that is, all directional derivatives exist at x_0) if and only if $\partial_f(x_0)$ is a single point which is then necessarily the G-derivative of f at x_0. Consequently f is G-differentiable at x_0 if and only if for every u, $f(x_0 + tu) + f(x_0 - tu) - 2f(x_0) = o(t)$ as $t \to 0$.

By considering $f(x) = \|x\|$ we recover the fact mentioned in section 6 that the norm is G-differentiable at x_0 in the unit sphere of X if and only if x_0 is a smooth point of the unit ball of X. It also follows that f is F-differentiable at x_0 if and only if $f(x_0+u)+f(x_0-u)-2f(x_0)=o(\|u\|)$ as $\|u\|\to 0$. **If a convex function f is F-differentiable in a neighborhood of x_0 then $D_f(x)$ is continuous there;** that is, F-differentiability of a convex function on an open set implies that it is C^1 there. Indeed, suppose that $x_n \to x_0$ and set $w^* := D_f(x_0)$; $u_n^* := D_f(x_n)$. Given $\epsilon > 0$ there is $\delta > 0$ so that $f(x_0+y)-f(x_0)-w^*(y) \le \epsilon\|y\|$ for $\|y\| \le \delta$. Pick y_n with $\|y_n\| = \delta$ and $(u_n^*-w^*)(y_n) \ge (\delta/2)\|u_n^*-w^*\|$ for all n. Then since $u_n^* = D_f(x_n)$, we have by the convexity of f that

$$\epsilon\delta + w^*(y_n) + f(x_0) \ge f(x_0+y_n) \ge u_n^*(y_n - x_n + x_0) + f(x_n)$$

or

$$\begin{aligned}(\delta/2)\ \|u_n^* - w^*\| &\le (u_n^* - w^*)(y_n) \\ &\le \epsilon\delta - f(x_n) + f(x_0) + \|u_n^*\| \cdot \|x_n - x_0\|.\end{aligned}$$

Since $\|u_n^*\|$ is bounded by the local Lipschitz constant of f, we conclude that $\limsup_n \|w^* - u_n^*\| \le 2\epsilon$.

Specializing again to $f(x) = \|x\|$, we see that the norm is G-differentiable at every nonzero x if and only if X is smooth (the norm is clearly never differentiable at zero); that is, for each x in the unit sphere of X there is a unique unit functional j_x of norm one which achieves its norm at x. The norm is then F-differentiable at every nonzero x if and only if the *duality mapping* j is a norm-to-norm continuous mapping. It follows from this and the Bishop-Phelps theorem that **if $\|\cdot\|$ is F-differentiable away from zero then the density character of X^* is equal to that of X.** In particular, X^* is separable if X is.

The space X is uniformly smooth if the norm is uniformly F-differentiable on the unit sphere; that is, if the limit $\lim_{t\to 0} t^{-1}(\|x+tu\|-\|x\|)$ exists uniformly in both x and u on the unit sphere.

The classical Gâteaux differentiability theorem for convex functions says: **A continuous convex function f on a separable Banach space X is G-differentiable on a dense G_δ set.** Indeed, if $\{x_n\}_{n=1}^{\infty}$ is dense in X then the set of G-differentiability of f is the set $\cap_{n,m} G_{n,m}$, where $G_{n,m}$ is the set of points x in X for which there exists $\delta > 0$ so that $f(x+\delta u_n)+f(x-\delta u_n)-2f(x) \le \delta/m$. It is readily verified that each $G_{n,m}$ is open and dense.

For Fréchet differentiability, the situation is much different even for norms. The norm of ℓ_1 is G-differentiable at any point all of whose coordinates are nonzero but is nowhere F-differentiable. This is typical in the sense that if X is separable and X^* is nonseparable then X admits an equivalent norm that is nowhere F-differentiable (see [11, Th. 106]). On the other hand: **If X^* is separable then every convex function f on X is F-differentiable on a dense G_δ.** That the set of points of F-differentiability is a G_δ is easy to check (and for this no separability assumption is needed); it is equal to $\cap_n G_n$ where G_n is the set of points x in X for which there exists $\delta > 0$ so that $\sup_{\|u\|\leq 1} f(x + \delta u) + f(x - \delta u) - 2f(x) \leq \delta/n$; the G_n's are obviously open. That each G_n is dense is not as obvious; see [11, Th. 114].

We now consider the differentiability of Lipschitz (or just locally Lipschitz) functions. The theorem of Lebesgue on the a.e. differentiability of Lipschitz functions on the line has a classical extension (called Rademacher's theorem) which says that every Lipschitz mapping between finite dimensional spaces is differentiable a.e. The proof goes over to show that the finite dimensional assumption on the range of the Lipschitz map can be replaced by the assumption that the range has the RNP. The main problem weakening the assumption on the domain space in Rademacher's theorem is that one must come up with an appropriate notion of a.e. or negligible set in an infinite dimensional space. It turns out that there are several nonequivalent ways to do this and we shall discuss one of them.

A Borel subset A of a separable Banach space X is said to be *Haar null* provided there exists a probability measure μ on the Borel subsets of X so that $\mu(x + A) = 0$ for every x in X. The Haar null sets form a σ-ring and they coincide with the usual Borel sets of Lebesgue measure zero if X is finite dimensional. In an infinite dimensional space X, every compact set A is Haar null since there is a direction so that every line in this direction cuts A in a set of linear measure zero and thus μ can be any probability measure supported on a line in this direction which is equivalent to linear Lebesgue measure.

Once one has the notion of Haar null set, the classical proof of Rademacher's theorem can be modified to prove: **Every locally Lipschitz mapping from a separable Banach space into a Banach space with the RNP is G-differentiable off a Haar null set.** For a fuller discussion and a discussion of other notions of negligible set, see [39] or [3].

This theorem does not hold for F-differentiability. Even ℓ_2 has an equivalent norm whose set of points of F-differentiability is a Haar null set (see [3, Ex. 6.46]). For locally Lipschitz functions one deep and useful result is: **Every real valued locally Lipschitz function on a space whose dual is separable is F-differentiable at a dense set of points** (again, see [39] or [3]). It is not known if this theorem remains true for Lipschitz functions taking values in the plane.

8 Finite dimensional Banach spaces

Since all finite dimensional spaces of the same dimension over the same scalar field are mutually isomorphic, for results on finite dimensional spaces to be meaningful they must be of a quantitative nature. The notion of Banach-Mazur distance is of central importance in this context. Evaluating or even estimating the distance between spaces is often hard since it generally is quite difficult to find an operator T for which $\|T\|\,\|T^{-1}\|$ is minimal or close to minimal.

We illustrate the computation of the Banach-Mazur distance by evaluating (or in some cases just giving the order of magnitude) the quantity $d(\ell_p^n, \ell_r^n)$. While relatively easy, even in this simple situation it is by no means trivial to calculate or even closely estimate the distance when p and r are on different sides of two. The topic of Banach-Mazur distances between finite dimensional space is treated in many more involved situations in [20].

Denote the formal identity mapping from ℓ_p^n to ℓ_r^n by $I_{p,r}$. It is trivial to check that $\|I_{p,r}\| = 1$ when $p \le r$ and $\|I_{p,r}\| = n^{1/r-1/p}$ when $p > r$. Consequently, if $1 \le p < r \le \infty$, then $d(\ell_p^n, \ell_r^n) \le n^{1/p-1/r}$. The simplest and most important case occurs when p (or r) is two. Either by induction from the case $n = 2$, where the following equality is just the parallelogram law, or by using the orthonormality of a Rademacher sequence and taking the vectors x_i to be in e.g. $L_2(0,1)$ and exchanging the order of integration over $[0,1]$ with the expectation, one sees that vectors $x_1, \ldots, x_n$ in a Hilbert space satisfy the identity

$$\mathbb{E}\,\|\sum_{i=1}^n \varepsilon_i x_i\|_2^2 = \sum_{i=1}^n \|x_i\|_2^2. \tag{15}$$

Suppose that $2 < r \le \infty$ and $T : \ell_r^n \to \ell_2^n$ is an isomorphism normalized to satisfy $\|T^{-1}\| = 1$ (so that $\|Tx\|_2 \ge \|x\|_r$ for all x). Denoting as usual the unit vectors basis as $\{e_i\}$, we see from (15) that $n \le \sum_{i=1}^n \|Te_i\|_2^2 = \mathbb{E}\,\|\sum_{i=1}^n \varepsilon_i T e_i\|_2^2 \le \max_\pm \|T\left(\sum_{i=1}^n \pm e_i\right)\|_2^2 \le \|T\|^2 n^{2/r}$, which is to say that $\|T\| \ge n^{1/2-1/r}$. From the equality $d(\ell_2^n, \ell_r^n) = n^{1/2-1/r}$ and the (multiplicative) triangle inequality for the Banach-Mazur distance one gets that $d(\ell_p^n, \ell_r^n) = n^{1/p-1/r}$ if $2 \le p \le r \le \infty$. By duality (or by repeating essentially the same argument) one gets the same is true if $1 \le p \le r \le 2$.

It turns out that when $p < 2 < r$ the identity operator $I_{p,r}$ does not give even a good approximation to the Banach-Mazur distance. For the upper estimate it is enough to work with complex scalars. This is because complex ℓ_p^n as a real space is isometric to the real space $(\sum_1^n \ell_2^2)_p$ which for all p has distance at most $\sqrt{2}$ to ℓ_p^{2n}. Using $d_{\mathbb{C}}(\cdot,\cdot)$ for the complex Banach-Mazur distance and

$d_{\mathbb{R}}(\cdot,\cdot)$ for the real distance, we thus have for even dimensions $d_{\mathbb{R}}(\ell_p^{2n},\ell_r^{2n}) \le 2d_{\mathbb{C}}(\ell_p^n,\ell_r^n)$ and for odd dimensions $d_{\mathbb{R}}(\ell_p^{2n+1},\ell_r^{2n+1}) \le 1 + 2d_{\mathbb{C}}(\ell_p^n,\ell_r^n)$.

The reason that the complex case is simpler is that there exists for every n a complex unitary matrix V_n of order n all of whose entries have absolute value $1/\sqrt{n}$ (take for V_n the matrix whose j,k entry is $n^{-1/2}\exp(2\pi ikj/n)$).

Denote the norm of an operator $\|T\|$ from ℓ_p^n to ℓ_r^n by $\|T\|_{p,r}$. Using the obvious identity $\|T\| = \max_i \|Te_i\|$ for operators T with domain ℓ_1^n, we have that $\|V_n\|_{1,\infty} = n^{-1/2}$. By considering V_n^* as the composition $I_{2,1}V_n^*I_{\infty,2}$ we see that $\|V_n^*\|_{\infty,1} \le n$. Since $V_n^* = V_n^{-1}$, it follows that $d(\ell_1^n,\ell_\infty^n) \le n^{-1/2}$. For the corresponding upper estimate for the distance for general $1 \le p < 2 < r \le \infty$, we need to use interpolation, which will be discussed in section 11 (but we use here only the most classical theorem). First we do the case $r = p^*$, where $1/p + 1/p^* = 1$. From the identities $\|V_n\|_{1,\infty} = n^{-1/2}$ and $\|V_n\|_{2,2} = 1$ interpolation gives us the inequality $\|V_n\|_{p,p^*} \le n^{1/2-1/p}$. Since $V_n^* = V_n^{-1}$ we see that

$$
\begin{aligned}
d(\ell_p^n,\ell_{p^*}^n) &\le \|V_n\|_{p,p^*}\|V_n^*\|_{p^*,p} \le n^{1/2-1/p}\|I\|_{2,p}\|V_n^*\|_{2,2}\|I\|_{p^*,2} \\
&= n^{1/2-1/p}n^{1/p-1/2}\cdot 1\cdot n^{1/p-1/2} = n^{1/p-1/2}.
\end{aligned}
$$

In the general case $1 \le p < 2 < r \le \infty$, by replacing the pair $\{p,r\}$ by $\{r^*,p^*\}$ if necessary, we can assume that $p^* \le r$. Then the triangle inequality for the Banach-Mazur distance gives us $d(\ell_p^n,\ell_r^n) \le n^{1/p-1/2}d(\ell_{p^*}^n,\ell_r^n) = n^{1/2-1/r}$. So for arbitrary $1 \le p < 2 < r \le \infty$ we have

$$
d(\ell_p^n,\ell_r^n) \le \max\{n^{1/p-1/2}, n^{1/2-1/r}\} \tag{16}
$$

in the case of complex scalars and something slightly worse in the case of real scalars (e.g. the same except that the right side of (16) is multiplied by three).

To show that the estimate (16) is accurate (up to a constant) one uses again an argument of averaging against a Rademacher sequence. The argument works in both the real and complex case. The inequality needed, called the *cotype 2 inequality* for L_p, will be discussed later in this section. It states that for $1 \le p \le 2$, any collection $x_1,\dots,x_n$ of vectors in an $L_p(\mu)$ space satisfies the inequality

$$
\left(\mathbb{E}\,\|\sum_{i=1}^n \varepsilon_i x_i\|_p^2\right)^{1/2} \ge A_p\left(\sum_{i=1}^n \|x_i\|_p^2\right)^{1/2}, \tag{17}
$$

where the expectation is with respect to the Rademacher sequence $\{\varepsilon_n\}_{n=1}^\infty$. Applying (17) to the image of the unit vector basis of ℓ_r^n, we get for $r > 2$ that

$d(\ell_r^n, E) \geq A_p n^{1/2-1/r}$ for every subspace E of an $L_p(\mu)$ space. Since $A_p \geq A_1$, this shows that the upper estimate of (16) is precise up to an absolute constant.

Since an n dimensional Banach space can be identified with $\mathbb{R}^n$ [or $\mathbb{C}^n$] with some symmetric [or balanced] closed bounded convex body as the unit ball, it is natural that geometric arguments often occur in the study of finite dimensional spaces. The notion of volume is of particular importance in this regard. Suppose that X and Y are both n dimensional Banach spaces, which we regard as $\mathbb{R}^n$ [or $\mathbb{C}^n$] with some norm, and $\alpha(\cdot)$ is a norm on the space $B(X,Y)$ of operators from X to Y. The natural algebraic dual of $B(X,Y)$ can be identified with $B(Y,X)$ under the *trace duality* $< S, T >:= \operatorname{trace} ST$ and we denote by $\alpha^*(\cdot)$ that norm on $B(Y,X)$ which makes $(B(Y,X), \alpha^*)$ the dual of $(B(X,Y), \alpha)$ under trace duality. By compactness of the unit ball of a finite dimensional space, there exists among all operators $T : X \to Y$ with $\alpha(T) \leq 1$ one which maximizes the volume of TB_X. Lewis' lemma states: **If** $\operatorname{vol}(TB_X)$ **achieves a maximum subject to the constraint** $\alpha(T) \leq 1$ **at** $T = T_0$**, then** T_0 **is invertible and** $\alpha^*(T_0^{-1}) = n$**.** This means $n^{-1}T_0^{-1}$ is a norm one functional on $(B(X,Y), \alpha)$ which achieves its norm at T_0.

Before giving the proof of Lewis' lemma, we mention one consequence of the idea of maximizing volume, Auerbach's lemma, which does not use Lewis' lemma: **If** Y **is an** n **dimensional Banach space then there is a basis** $\{x_k, x_k^*\}_{k=1}^n$ **so that for each** k**,** $\|x_k\| = 1 = \|x_k^*\|$**.** We take $X = \ell_1^n$ and let $\alpha(\cdot)$ be the operator norm: Having gotten $T_0 : \ell_1^n \to Y$ to maximize the volume of the image of the ℓ_1^n ball as above, it is obvious that the basis $\{x_k, x_k^*\}_{k=1}^n$ works, where $x_k := Te_k$, $x_k^*(x) := \det(T_0)^{-1}\det(T_k)$, and $T_k : \ell_1^n \to Y$ is defined by letting $T_k e_i$ be x_i when $i \neq k$ and $T_k e_k = x$.

We turn to the proof of Lewis' lemma. Since the volume of TC for any measurable set is a constant multiple of $|\det(T)| \operatorname{vol}(C)$, for any operator $S : X \to Y$ we have

$$\left|\det\left(\frac{T_0 + S}{\alpha(T_0 + S)}\right)\right| \leq |\det(T_0)|. \tag{18}$$

Certainly T_0 must be invertible, so by dividing (18) by $|\det(T_0)|$ we can rewrite (18) as

$$|\det(I_X + T_0^{-1}S)| \leq \alpha(T_0 + S)^n. \tag{19}$$

Since $\alpha(T_0) \leq 1$ we get from (19) that for all $\epsilon > 0$ and all operators $S : X \to Y$:

$$|\det(I_X + \epsilon T_0^{-1}S)| \leq (1 + \epsilon\alpha(S))^n. \tag{20}$$

Obviously $\det(I_X+\epsilon T) = 1+\epsilon(\operatorname{trace} T)+o(\epsilon)$ as $\epsilon \to 0$ (that is, the trace functional is the derivative of the determinant at the identity operator), so from (20) we conclude that $\operatorname{trace} T_0^{-1}S \le n\alpha(S)$ for all operators $S : X \to Y$ which in view of the trace duality between $\alpha(\cdot)$ and $\alpha^*(\cdot)$ means that $\alpha^*(T_0^{-1}) \le n$.

The first application of Lewis' lemma comes from applying it when $\alpha(\cdot)$ is the operator norm. (This case, known before the lemma, see e.g. [20, p. 137], in fact motivated the discovery of the lemma.) The norm $\mathcal{N}(\cdot)$ on $B(Y,X)$ which is dual to the operator norm on $B(X,Y)$ for finite dimensional X and Y is called the *nuclear norm.* To identify the nuclear norm on $B(Y,X)$, observe first that if $\|y^*\| = 1 = \|x\|$ with y^* in Y^* and x in X, then $\mathcal{N}(y^* \otimes x) = 1$, where $y^* \otimes x$ is the rank one operator from Y to X defined by $(y^* \otimes x)(z) = y^*(z)x$. Next note that for any operator $T : X \to Y$,

$$\begin{aligned}\|T\| &= \sup\{y^*(Tx) : \|y^*\| = 1 = \|x\|\} \\ &= \sup\{\operatorname{trace}(y^* \otimes x)T : \mathcal{N}(y^* \otimes x) = 1\}.\end{aligned}$$

Applying the bipolar theorem we conclude that the unit ball of $(B(Y,X), \mathcal{N}(\cdot))$ is the convex hull of the closed set $W := \{y^* \otimes x : \|y^*\| = 1 = \|x\|\}$. From this description of the unit ball of $(B(Y,X), \mathcal{N}(\cdot))$ one obtains that for any S in $B(Y,X)$, $\mathcal{N}(S) = \min \sum_{k=1}^N \|y_k^*\| \cdot \|x_k\|$, where the minimum is over all representations $S = \sum_{k=1}^N y_k^* \otimes x_k$. Caratheodory's theorem says that the minimum occurs already with N at most one plus the dimension of $B(Y,X)$ as a vector space over the reals; that is, $N \le n^2 + 1$ in the case of real scalars and $N \le 2n^2 + 1$ when the scalars are complex. (By being slighty more careful before applying Caratheodory's theorem, one obtains the best estimates $N \le n^2$ and $N \le 2n^2$; see [20, 8.6]). Having done these preliminaries, we can state a nice geometric reformulation of the case $\alpha(\cdot) = \|\cdot\|$ in Lewis' lemma. Again identify both spaces X and Y with $\mathbb{R}^n$ or $\mathbb{C}^n$. When one convex body contains another, a point x is called a *contact point* of the bodies if it is in the intersection of their boundaries (so if the bodies are unit balls associated with two norms, a contact point is a point in the intersection of the two unit spheres). **Assume that $B_X \subset B_Y$ and $\operatorname{vol}(B_X) \ge \operatorname{vol}(TB_X)$ for every operator T on $\mathbb{R}^n$ $[\mathbb{C}^n]$ for which $TB_X \subset B_Y$. Then there exist contact points $x_1, \dots, x_N$ of B_X and B_Y and contact points $x_1^*, \dots, x_N^*$ of B_{X^*} and B_{Y^*} and $c_k \ge 0$ so that $I = \sum_{k=1}^N c_k x_k^* \otimes x_k$. Also, $N \le n^2$ in the real case and $N \le 2n^2$ in the complex case.**

The most important applications of Lewis' lemma occur when $X = \ell_2^n$. In this case observe that the operator T_0 is unique up to a unitary operator on ℓ_2^n. Indeed, suppose that T_1 is another operator from ℓ_2^n for which $\alpha(T_1) = 1$ and $\alpha^*(T_1^{-1}) = n$. By preceding T_1 with a unitary we can assume that $T_0^{-1}T_1$ is a positive operator D with eigenvalues $\alpha_1 \ge \ldots \ge \alpha_n > 0$. Then $\sum_{k=1}^n \alpha_k = \operatorname{trace} D \le \alpha^*(T_0^{-1})\alpha(T_1) \le n$ while $\sum_{k=1}^n 1/\alpha_k = \operatorname{trace} D^{-1} \le \alpha^*(T_1^{-1})\alpha(T_0) \le$

n, so that $\alpha_k = 1$ for all k. Specializing to the case of the operator norm, we obtain a classical result called John's theorem: **Let X be $\mathbb{R}^n$ or $\mathbb{C}^n$ under some norm $\|\cdot\|$. Then B_X contains a unique ellipsoid of maximal volume. This ellipsoid is the Euclidean ball $B := B_{\ell_2^n}$ if and only if there exist contact points $\{u_k\}_{k=1}^N$ between B and B_X and $c_k > 0$ so that for all x in X,**

$$x = \sum_{k=1}^{N} c_k \langle u_k, x \rangle u_k. \tag{21}$$

Moreover, $N \le n(n+1)/2$ in the real case and $N \le n^2$ in the complex case. To see that John's theorem follows from the geometric version of Lewis' lemma, write the identity on X as $I_X = \sum_{k=1}^N c_k x_k^* \otimes x_k$ with the x_k contact points between B and B_X, the x_k^* contact points between B and B_{X^*}, and the c_k positive real numbers which sum to n. Here B_{X^*} is identified with those y in X for which $|\langle y, x \rangle| \le 1$ for all x in B_X and the duality pairing is given by the usual inner product. By taking traces we see that $\langle x_k^*, x_k \rangle = 1$ for all k and hence, since the Euclidean norm of x_k^* and x_k are both one, that x_k^* is the functional $\langle \cdot, x_k \rangle$. This gives John's theorem except for the improved estimate on N, which is easy but we omit (see [20, 8.6]).

An important consequence of John's theorem for Banach space theory is: **If X is n dimensional then $d(X, \ell_2^n) \le \sqrt{n}$.** Indeed, by (21) since $\langle u_k, \cdot \rangle$ is a norm one functional on X, we have for all x in X that $\|x\|_2^2 = \langle x, x \rangle = \sum_{k=1}^N c_k |\langle u_k, x \rangle|^2 \le \sum_{k=1}^N c_k = n$. This says that $B_X \subset \sqrt{n} B$.

For general spaces of dimension n we thus have $d(X, Y) \le d(X, \ell_2^n) d(\ell_2^n, Y) \le n$. It turns out that this estimate is precise up to a constant independent of n. There exist for each n spaces X_n and Y_n of dimension n so that $\inf d(X_n, Y_n)/n > 0$. If one puts $2n$ pairs of symmetric points on the unit sphere of ℓ_2^n, where the points are chosen independently and are distributed uniformly on that sphere, and then takes the symmetric convex hull of the union of these points with the unit vector basis, one obtains a unit ball for a (random) space. If one takes two of these random spaces, then with big probability (that is, with probability tending to one as $n \to \infty$) the Banach-Mazur distance between the two spaces exceeds δn for some constant $\delta > 0$ independent of n. Although this construction is easy to describe, the computations are delicate and the reader is referred to [20, 38.1] for details. The probabilistic approach has many other applications. For example, from the estimate $d(X, \ell_2^n) \le \sqrt{n}$ it follows immediately that an n dimensional space has a basis with basis constant at most $\sqrt{n}$. Spaces of the type X_n mentioned above exist (even with big probability) so that any basis in X_n has basis constant at least $\delta\sqrt{n}$. A discussion of these examples and their applications to infinite dimensional theory is given in [36].

The maximal volume ellipsoid appears also in another geometrical result which is a weak version of the classical Dvoretzky-Rogers lemma (see [28]). Again let X be $\mathbb{R}^n$ or $\mathbb{C}^n$ under some norm for which the Euclidean ball is the ellipsoid of maximum volume contained in B_X. Then **there exist** $[n/2]$ **orthonormal vectors** x_k **so that** $\|x_k\|_X \geq 1/2$ **for all** k**.** Rather than deduce this from John's theorem, we prove it directly. Construct $\{x_k\}_{k=1}^n$ inductively so that $\|x_{k+1}\|_X$ is maximized subject to the constraints that $\|x\|_2 = 1$ and x is orthogonal to $x_1, \ldots, x_k$. Fix any $k < n$, set $\beta = \|x_k\|_X$, and let X_k be the span of $x_1, \ldots, x_k$. Define an ellipsoid D by $D := \{x + y : x \in X_k; y \in X_k^{\perp}; 2\|x\|_2^2 + 2\beta^2\|y\|_2^2 \leq 1\}$. One checks easily that $D \subset B_X$. By comparing the volume of D to that of B one gets that $2^{-n/2}\beta^{-(n-k+1)} \leq 1$ which gives the estimate $\|x_k\|_X \geq 2^{-n/2(n-k+1)}$.

A geometric interpretation of the Dvoretzky-Rogers lemma is that whenever X has dimension $2n$ there is a subspace Y of dimension n so that in an appropriate coordinate system the unit ball of Y contains the Euclidean ball and is contained in twice the unit cube $B_{\ell_\infty^n}$. Since ℓ_∞^n is 1-injective, this implies that the identity $I_{2,\infty} : \ell_2^n \to \ell_\infty^n$ can be written as a product $T_1 T_2$ with $\|T_2 : \ell_2^n \to X\| = 1$ and $\|T_1 : X \to \ell_\infty^n\| \leq 2$.

A result which goes much further is Dvoretzky's theorem, which says that every infinite dimensional Banach spaces contains for every n subspaces whose Banach-Mazur distance to ℓ_2^n is arbitrarily close to one. The finite dimensional quantitative version of Dvoretzky's theorem from which the infinite dimensional statement above follows immediately says: **For every** k **and** $\epsilon > 0$ **there exists** $n_0 = n_0(\epsilon, k)$ **so that if dim**$(X) \geq n_0$ **then** X **contains a subspace** Y **with** $d(Y, \ell_2^k) < 1+\epsilon$**.** There are known good estimates on $n_0(\epsilon, k)$ (and even better estimates for special classes of spaces). For general spaces $n_0(\epsilon, k) \leq \exp(\alpha k/\epsilon^2)$ for some constant α. Dvoretzky's theorem is treated in detail in several books, including [9, 19.1], [17, 4.3], [16, I.5.8]. An exposition of Dvoretzky's theorem and related results is given in [28]. There are many proofs of Dvoretzky's theorem and in most of them the Dvoretzky-Rogers lemma is the first step.

From Dvoretzky's theorem and the technique for constructing basic sequences discussed in section 3 one gets in any infinite dimensional Banach space a basic sequence $\{x_n\}_{n=1}^\infty$ so that for each k, $\{x_j\}_{j=2^k+1}^{2^{k+1}}$ is $1/k$-equivalent to an orthonormal basis in a Hilbert space. An easy consequence of this is that for every square summable sequence $\{\alpha_n\}_{n=1}^\infty$, of scalars, every infinite dimensional Banach space contains an unconditionally convergent series $\sum_n y_n$ such that for each n, $\|y_n\| = |\alpha_n|$. One can also easily deduce this from the simple Dvoretzky-Rogers lemma; in fact, historically it was this application that motivated the discovery of the Dvoretzky-Rogers lemma.

A result related to Dvoretzky's theorem is Krivine's theorem: **If** $\{x_n\}_{n=1}^\infty$ **is a basic sequence in a Banach space, then there exists** $1 \leq p \leq \infty$ **so**

that for every k and every $\epsilon > 0$ there is a block basis of $\{x_n\}_{n=1}^{\infty}$ of length k which is $(1+\epsilon)$-equivalent to the unit vector basis of ℓ_p^k. This statement can be "localized" to obtain a statement about all sufficiently long finite basic sequences having a specified basis constant (or even biorthogonality constant). We do not do so here for two reasons: (1) The finite statement follows formally from the infinite statement via a localization technique that will be mentioned in section 9, and (2) Unlike the situation in Dvoretzky's theorem, the quantitative estimates in the finite statement are so poor that there are no known consequences of them beyond what can be deduced formally from the infinite version of Krivine's theorem stated above. For a proof of Krivine's theorem see [16, Ch. 12] or [3, Ch. 12].

In the setting of Banach lattices, Krivine's theorem yields that for every infinite dimensional Banach lattice there exists $1 \le p \le \infty$ so that for every k and every $\epsilon > 0$ there is a lattice isomorphism T from ℓ_p^n into X with $\|T\| \, \|T^{-1}\| < 1+\epsilon$.

From Krivine's theorem and the fact discussed in section 4 that ℓ_2 embeds isometrically into $L_p(0,1)$ it is easy to deduce Dvoretzky's theorem. This is not the recommended route to Dvoretzky's theorem as it is difficult to navigate through Krivine's theorem. More importantly, the tight quantitative estimates obtainable from direct proofs of Dvoretzky's theorem have many applications.

If we apply Krivine's theorem to a basic sequence which is equivalent to the unit vector basis for ℓ_r we see immediately that the only p that is obtainable is $p = r$. Thus by applying Krivine's theorem to a disjoint sequence in an infinite dimensional $L_p(\mu)$ space we infer that any equivalent renorming of $L_p(\mu)$ contains for every k and $\epsilon > 0$ a subspace whose Banach-Mazur distance to ℓ_p^k is less than $1+\epsilon$. This implies that an infinite dimensional $L_p(\mu)$ space cannot be given an equivalent norm which has a better modulus of convexity or smoothness than its natural norm.

Two notions that are very important for both the finite dimensional and infinite dimensional theories are that of type and cotype. A Banach space X is said to have *type* p provided there is a constant C so that for every sequence $x_1, \dots, x_n$ in X,

$$\left(\mathbb{E}\,\|\sum_{i=1}^{n} \varepsilon_i x_i\|^2\right)^{1/2} \le C \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \tag{22}$$

(with the usual modification for $p = \infty$). The expectation is with respect to the Rademacher sequence $\{\varepsilon_n\}_{n=1}^{\infty}$. Similarly, X is said to have *cotype* q

provided there is a constant C so that for every sequence $x_1, \dots, x_n$ in X,

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq C \left(\mathbb{E} \| \sum_{i=1}^{n} \varepsilon_i x_i \|^2 \right)^{1/2}. \tag{23}$$

The best constants in (22) and (23) are denoted by $T_p(X)$ and $C_q(X)$. By the parallelogram identity characterization of Hilbert space, a space X is isometric to a Hilbert space if and only if $T_2(X) = 1 = C_2(X)$. Even a one dimensional space does not have type p for any $p > 2$ or cotype q for any $q < 2$. For every space X, $T_1(X) = 1$ and $C_\infty(X) = 1$ by convexity of the norm. As functions of p and q, $T_p(X)$ is nondecreasing and $C_p(X)$ is nonincreasing. The inequalities $T_p(X) \leq d(X,Y)T_p(Y)$ and $C_q(X) \leq d(X,Y)C_q(Y)$ are evident, so for an n dimensional X, in the range $1 \leq p \leq 2 \leq q \leq \infty$ the quantities $T_p(X)$ and $T_q(X)$ cannot exceed $\sqrt{n}$ by the distance estimate of X to ℓ_2^n.

A Banach lattice X which is p-convex and q-concave has type $p_0 := p \wedge 2$ and cotype $q_0 := q \vee 2$. The type assertion follows from (10) and monotonicity of $M^{(t)}$ and $M_{(s)}$:

$$\begin{aligned}
\left(\mathbb{E} \| \textstyle\sum_n \varepsilon_n x_n \|^2\right)^{1/2} &\leq M_{(q_0)}(X) B_{q_0} \| \left(\textstyle\sum_n |x_n|^2\right)^{1/2} \| \\
&\leq M_{(q)}(X) B_{q_0} M^{(p)}(X) \left(\textstyle\sum_n \|x_n\|^{p_0}\right)^{1/p_0}.
\end{aligned}$$

For the lattice cotype assumption we need Khintchine's inequality (1) as well as monotonicity of the convexity and concavity parameters:

$$\begin{aligned}
&\left(\textstyle\sum_n \|x_n\|^{q_0}\right)^{1/q_0} \leq M_{(q_0)}(X) \| \left(\textstyle\sum_n |x_n|^{q_0}\right)^{1/q_0} \| \\
&\quad \leq M_{(q)}(X) \| \left(\textstyle\sum_n |x_n|^2\right)^{1/2} \| \leq M_{(q)}(X) A_{p_0}^{-1} \| \left(\mathbb{E} | \textstyle\sum_n \varepsilon_n x_n|^{p_0}\right)^{1/p_0} \| \\
&\quad \leq M_{(q)}(X) A_{p_0}^{-1} M^{(p_0)}(X) \left(\mathbb{E} \| \textstyle\sum_n \varepsilon_n x_n\|^{p_0}\right)^{1/p_0} \\
&\quad \leq M_{(q)}(X) A_{p_0}^{-1} M^{(p)}(X) \left(\mathbb{E} \| \textstyle\sum_n \varepsilon_n x_n\|^2\right)^{1/2}.
\end{aligned}$$

Specializing to $L_p(\mu)$ spaces, we see that for $1 \leq p \leq 2$, $T_p(L_p) = 1$ and $C_2(L_p) \leq A_p^{-1} (\leq A_1^{-1})$, while for $2 \leq p < \infty$, $T_2(L_p) \leq B_p$ and $C_p(L_p) = 1$. Notice that this implies that for $1 \leq r < p \leq 2$, ℓ_r is not isomorphic to a subspace of any $L_p(\mu)$ space.

By considering a disjoint sequence in the space, we see that an infinite dimensional $L_p(\mu)$ space with $1 \leq p \leq 2$ cannot have type better than p. Also, since infinite dimensional $L_\infty(\mu)$ spaces are universal for separable spaces, they do not have type p for any $p > 1$. Similarly, an infinite dimensional $L_q(\mu)$ space for $2 \leq q \leq \infty$ does not have cotype smaller than q.

If $\{x_n\}_{n=1}^\infty$ is a seminormalized unconditionally basic sequence in $L_q(\mu)$, $2 < q < \infty$, with μ a probability measure and $\inf \|x_n\|_2 > 0$, then since $L_q(\mu)$ has type 2 and $L_2(\mu)$ has cotype 2 and $\|\cdot\|_2 \le \|\cdot\|_q$ we infer that $\{x_n\}_{n=1}^\infty$ is, in $L_q(\mu)$, equivalent to the unit vector basis of ℓ_2. Thus such a sequence cannot have dense linear span in $L_q(\mu)$. In particular, the trigonometric system is not unconditional in $L_q(0,1)$ for $2 < q \le \infty$ and, by duality, also not in $L_p(0,1)$, $1 \le p < 2$.

If $\{x_n\}_{n=1}^\infty$ is a seminormalized sequence in $L_q(0,1)$, $2 < q < \infty$, and $\inf \|x_n\|_2 > 0$, then we saw in section 4 that $\{x_n\}_{n=1}^\infty$ has a subsequence which is equivalent to the unit vector basis for ℓ_q. Since $L_p(0,1)$, $1 < p < \infty$, has an unconditional basis (the Haar system), it follows from the discussion in section 3 that every seminormalized weakly null sequence in $L_p(0,1)$ has an unconditionally basis subsequence. Combining these comments with those in the previous paragraph, and using the fact that every separable subspace of $L_p(\mu)$ embeds isometrically into $L_p(0,1)$, we conclude that **every seminormalized weakly null sequence in $L_q(0,1)$, $2 < q < \infty$, has a subsequence which is equivalent to the unit vector basis for either ℓ_2 or ℓ_q.**

The second moment in the definitions of type and cotype can be replaced with any other moments (but of course then the constants change in (22) and (23)). This is a consequence of the Kahane-Khintchine inequality, which says that for every $0 < p < \infty$ there are constants A_p and B_p so that for every sequence $x_1, \dots, x_n$ in a Banach space,

$$A_p \left(\mathbb{E} \| \sum_{i=1}^n \varepsilon_i x_i \|^2 \right)^{1/2} \le \left(\mathbb{E} \| \sum_{i=1}^n \varepsilon_i x_i \|^p \right)^{1/p} \le B_p \left(\mathbb{E} \| \sum_{i=1}^n \varepsilon_i x_i \|^2 \right)^{1/2} \quad (24)$$

Here the expectation is with respect to the Rademacher sequence $\{\varepsilon_n\}_{n=1}^\infty$. The inequality (24) follows via extrapolation from the following *hypercontractivity inequality:* for every $2 < p < \infty$ there is a constant $\sigma_p > 0$ so that for every sequence $x, x_1, \dots, x_n$ in a Banach space,

$$\left(\mathbb{E} \|x + \sum_{i=1}^n \varepsilon_i \sigma_p x_i \|^p \right)^{1/p} \le \left(\mathbb{E} \|x + \sum_{i=1}^n \varepsilon_i x_i \|^2 \right)^{1/2}. \quad (25)$$

To prove (25), first note that there is $\sigma_p > 0$ so that for all $0 < t \le 1$,

$$\left(\frac{|1 + \sigma_p t|^p + |1 - \sigma_p t|^p}{2} \right)^{1/p} \le (1 + t^2)^{1/2}. \quad (26)$$

This follows, for example, from L'Hôpital's rule. The best constant is $\sigma_p =$

$(p-1)^{-1/2}$ (see [15, 1.e.14]). Secondly, deduce a vector valued version of (26):

$$(\mathbb{E}\,\|x+\varepsilon_1\sigma_p y\|^p)^{1/p} \le (\mathbb{E}\,\|x+\varepsilon_1 y\|^2)^{1/2}. \tag{27}$$

One can assume that $\|x+y\|+\|x-y\|=2$ with $t := \frac{\|x+y\|-\|x-y\|}{2} \ge 0$. Then for $\varepsilon = \pm 1$,

$$\|x+\sigma_p\varepsilon y\| \le \frac{1+\sigma_p}{2}\|x+\varepsilon y\| + \frac{1-\sigma_p}{2}\|x-\varepsilon y\| = 1+\varepsilon\sigma_p t.$$

Therefore

$$(\mathbb{E}\,\|x+\varepsilon_1\sigma_p y\|^p)^{1/p} \le (\mathbb{E}\,(1+\sigma_p\varepsilon_1 t)^p)^{1/p} \le \sqrt{1+t^2} = \left(\mathbb{E}\,\|x+\varepsilon_1 y\|^2\right)^{1/2}.$$

Finally, iterate (27) to obtain (25). Formally, for $k=1,\dots,n$ define $S_k := \sum_{i=1}^k \varepsilon_i x_i$; we need to show that $(\mathbb{E}\|x+\sigma_p S_k\|^p)^{1/p} \le (\mathbb{E}\|x+S_k\|^2)^{1/2}$. For $k=1$ this is (27). Assume that the desired inequality is true for k and let $\mathbb{E}_{k+1}$ be conditional expectation with respect to ε_{k+1} and $\mathbb{E}^k$ conditional expectation with respect to $\varepsilon_1,\dots,\varepsilon_k$. Then:

$$\begin{aligned}(\mathbb{E}\|x+\sigma_p S_{k+1}\ \|^p)^{2/p} &= \left(\mathbb{E}^k\mathbb{E}_{k+1}\|(x+\sigma_p S_k)+\varepsilon_{k+1}\sigma_p x_{k+1}\|^p\right)^{2/p}\\ &\le \left(\mathbb{E}^k(\mathbb{E}_{k+1}\|(x+\sigma_p S_k)+\varepsilon_{k+1}x_{k+1}\|^2)^{p/2}\right)^{2/p} \quad \text{by (27)}\\ &\le \mathbb{E}_{k+1}\left(\mathbb{E}^k\|(x+\varepsilon_{k+1}x_{k+1})+\sigma_p S_k\|^p\right)^{2/p} \quad \text{by Minkowski}\\ &\le \mathbb{E}_{k+1}\mathbb{E}^k\|(x+\varepsilon_{k+1}x_{k+1})+S_k\|^2 = \mathbb{E}\|x+S_{k+1}\|^2.\end{aligned}$$

The theory of type and cotype is quite extensive. In the rest of this section we mention without proof a sample of results in this theory. Expositions of this theory can be found in several books; in particular, [15], [16], [17], [20]; and see the article [38].

We already mentioned that a space isomorphic to a Hilbert space is of type 2 and cotype 2. The converse is true and this is one of the basic isomorphic characterizations of Hilbert space.

There is a connection between the theory of cotype and Dvoretzky's theorem. If X is an n dimensional space then X has a subspace Y of dimension $k \ge n^{2/p}/N(p, C_p(X), \epsilon)$ with $d(Y, \ell_2^k) \le 1+\epsilon$. In particular, if X is a subspace of $L_1(\mu)$ then the k above is proportional to n, with a constant depending just on ϵ. See [17, 4.15] or [16, 5.3].

A connection between type and cotype theory and Krivine's theorem is provided by the Maurey-Pisier theorem: **If X is an infinite dimensional**

Banach space and $p_0 := \sup\{p : X \text{ has type } p\}$ **and** $q_0 := \inf\{q : X \text{ has cotype } q\}$**, then for any** $\epsilon > 0$ **and any** k **there are subspaces** Y_k **and** Z_k **of** X **so that** $d(Y_k, \ell_{p_0}^k) \leq 1+\epsilon$**, and** $d(Z_k, \ell_{p_0}^k) \leq 1+\epsilon$**.** A proof of this is given in [16, II.13.2].

The special cases $p_0 = 1$ and $q_0 = \infty$ of the Maurey-Pisier theorem are especially important (and much easier to prove; they do not require Krivine's theorem). If X does not have type p for any $p > 1$, then X contains almost isometric copies of ℓ_1^n (the converse is true and obvious). Spaces which do not contain almost isometric copies of ℓ_1^n for all n arose first in the study of probability in Banach spaces and are sometimes called *B-convex* (see [34]).

Similarly, a space does not have finite cotype if and only if it contains for every n arbitrarily close copies of ℓ_∞^n. Since ℓ_1^k is a subspace of $\ell_\infty^{2^k}$ (isometrically in the case of real scalars and up to constant in the complex case), we deduce that if X is B-convex then it has cotype q for some $q < \infty$. Since a uniformly convex space cannot have subspaces of arbitrary dimension which are arbitrarily close to ℓ_1^n, it follows that a uniformly convex space is B-convex; that is, has nontrivial type and cotype.

The relation of B-convexity to reflexivity is not so simple but has been clarified. A B-convex space need not be reflexive–there is even a nonreflexive space of type 2. Though nonreflexive Banach spaces need not contain almost isometric copies of ℓ_1^n, they do contain configurations of the following type for every n and $\epsilon > 0$: norm one vectors $x_1, \dots, x_n$ so that for every $k < n$, $\|x_1 + \ldots + x_k - (x_{k+1} + \ldots + x_n)\| \geq n - \epsilon$. In particular, taking $n = 2$ we see that: **Every nonreflexive space contains real subspaces arbitrarily close to real** ℓ_1^2**.** That is, a real space whose unit ball does not have a two dimensional section arbitrarily close to a square must be reflexive. See [2, 4.III] for proofs of these results.

We next discuss the duality theory of type and cotype. It is simple that **if** X **has type** p **then** X^* **has cotype** p^* and $C_{p^*}(X^*) \leq T_p(X)$. Indeed, given $x_1^*, \dots, x_n^*$ in X^* and $\epsilon > 0$, take $x_1, \dots, x_n$ in X so that $\sum_{i=1}^n x_i^*(x_i) \geq (1-\epsilon)\left(\sum_{i=1}^n \|x_i^*\|^{p^*}\right)^{1/p^*} \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$. Using (4), we get

$$\begin{aligned}
\sum_{i=1}^n x_i^*(x_i) &= \mathbb{E}\left(\sum_{i=1}^n \varepsilon_i x_i^*\right)\left(\sum_{i=1}^n \varepsilon_i x_i\right) \\
&\leq \left(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i^*\|^2\right)^{1/2} \left(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i\|^2\right)^{1/2} \\
&\leq T_p(X)\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \left(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i^*\|^2\right)^{1/2}.
\end{aligned}$$

In the opposite direction this does not work: L_1 has cotype 2 but L_∞ does not have type 2. Since $L_2(X^*)$ is a subspace of $L_2(X)^*$, we have

$$\left(\mathbb{E}\,\|\sum_{i=1}^{n}\varepsilon_i x_i^*\|^2\right)^{1/2} = \sup\{\mathbb{E}\,\left(\sum_{i=1}^{n}\varepsilon_i x_i^*\right) f : \|f\|_{L_2(X)} \leq 1\}.$$

The problem is the passage from a general f to a function of the form $\sum_{i=1}^{n}\varepsilon_i x_i$. This would be possible if the Rademacher projection $\tilde{P} := P \otimes I_X$ from $L_2(X)$ into itself would be bounded, where P is the orthogonal projection onto the closed linear span of the Rademacher sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ (a space X for which $\tilde{P}$ is bounded is said to be K-*convex*). $\tilde{P}$ is defined explicitly for f_i in $L_2(\mu)$ and x_i in X by $\tilde{P}(\sum_{i=1}^{n} f_i x_i) = \sum_{i=1}^{n}(Pf_i)x_i$. It is a deep fact (see [16, II.14], [17, 2.4], and [38]) that: **X is K-convex if and only if X is B-convex.** Thus the presence of copies of ℓ_1^n is exactly the factor which hinders a clean duality between type and cotype. **If X is B-convex and of cotype p^* then X^* is of type p.**

For finite dimensional X it is important to estimate the norm of the Rademacher projection (called the K-*convexity constant* of X and usually denoted by $K(X)$). A useful estimate (which is valid also for isomorphs of infinite dimensional Hilbert space) is $K(X) \leq K(1 + \log d(X, H))$ for some constant K, where H is a Hilbert space with the same dimension as X. When $\dim(X) = n > 1$, the distance estimate from John's theorem gives that $K(X) \leq K_1 \log n$. These estimates are sharp up to the values of the constants K and K_1. For a discussion see [20, pp. 86-92], [17], [16, II.14], or [38].

9 Local structure of infinite dimensional spaces

In this section we describe results and techniques whose purpose is to relate the structure of an infinite dimensional Banach space with the structure of its finite dimensional subspaces. We will be particularly interested in properties of a space which depend only on its family of finite dimensional subspaces and not on the way these finite dimensional spaces are "glued together" to form the infinite dimensional space.

A Banach space X is said to be λ-*representable* in a Banach space Y if for every finite dimensional subspace E of X and every $\epsilon > 0$ there is a finite dimensional subspace F of Y with $d(E, F) \leq \lambda + \epsilon$. If this holds with $\lambda = 1$ we say that X is *finitely representable* in Y. In this terminology we can rephrase some results discussed in the previous section as follows: The space ℓ_2 is finitely representable in every infinite dimensional Banach space Y (Dvoretzky's theorem). A Banach space Y has cotype q for some $q < \infty$ if and only if c_0 is not finitely representable in Y. A Banach space Y has type p for some $p > 1$ if and only if ℓ_1 is not finitely representable in Y. It is a trivial fact that every Banach space is finitely representable in c_0.

For every Banach space X its second dual X^{**} is finitely representable in X. In fact the following stronger statement, called the *principle of local reflexivity*, is true: **Let E be a finite dimensional subspace of X^{**}, F a finite dimensional subspace of X^* and $\epsilon > 0$. Then there is an operator T from E into $TE \subset X$ so that $\|T\|\ \|T^{-1}\| \leq 1+\epsilon$, $T_{|E\cap X}$ is the identity and $x^{**}(x^*) = x^*(Tx^{**})$ for every $x^* \in F$ and $x^{**} \in E$.**

Before giving a proof of the principle itself let us make some simple observations. If $S : Y \to Z$ is an operator with closed range then so is $S+R$ if R is a finite rank operator. Also if S is as above and there is a $y^{**} \in Y^{**}$ such that $S^{**}y^{**} \in Z$ (i.e. the canonical image of Z in Z^{**}) then there is for every $\delta > 0$ a $y \in Y$ such that $\|y\| \leq \|y^{**}\|(1+\delta)$ and $S^{**}y^{**} = Sy$. (This reduces immediately to the case where S is a quotient map and in this case it is obvious). Another simple observation is the following: for every ϵ there is a $\delta(\epsilon)$ so that if $(1+\delta)^{-1} \leq \|Sy\| \leq 1+\delta$ for every y in a δ-net in the unit sphere of Y, and some operator S, then $(1+\epsilon)^{-1}\|y\| \leq \|Sy\| \leq (1+\epsilon)\|y\|$ for all $y \in Y$.

We pass to the proof of the principle of local reflexivity.

Let $\epsilon > 0$ and let $\delta = \delta(\epsilon)$ be as above. Let E and F be finite dimensional subspaces of X^{**}, respectively, X^*. We pick $\{v_j^*\}_{j=1}^m$ in the unit ball of X^* so that the set contains an algebraic basis of F and so that $\|x^{**}\| \leq (1+\delta)\max\limits_j |x^{**}(v_j^*)|$ for every $x^{**} \in E$. Let $\{w_i^{**}\}_{i=1}^n$ be a δ-net in the unit sphere of E so that $\{w_i^{**}\}_{i=1}^k$ is a basis of $E \cap X$ and $\{w_i^{**}\}_{i=1}^r$ is a basis of E for some $k \leq r < n$. Write $w_i^{**} = \sum\limits_{h=1}^r \lambda_{i,h} w_h^{**}$ for $r < i \leq n$. For $r < i \leq n$ put $\mu_{i,h} = \lambda_{i,h}$ if $h \leq r$, $\mu_{ii} = -1$, and $\mu_{i,h} = 0$ for $r < h \neq i$. Consider the operator $S : (\overbrace{X \oplus \cdots \oplus X}^{n})_\infty \to (\overbrace{X \oplus \cdots \oplus X}^{n-r+k})_\infty$ defined by

$$S(x_1, \ldots, x_n) = \left(x_1, \ldots, x_k, \sum_{h=1}^n \mu_{r+1,h} x_h, \ldots, \sum_{h=1}^n \mu_{n,h} x_h \right).$$

By the choice of $\mu_{i,h}$ for $h > r$ it follows that S is onto. Hence the operator $\widetilde{S} : (\overbrace{X \oplus \cdots \oplus X}^{n})_\infty \to (\overbrace{X \oplus \cdots \oplus X}^{n-r+h} \oplus \mathbb{R}^{nm})_\infty$ defined by $\widetilde{S}(x_1, \ldots, x_n) = (S(x_1, \ldots, x_n), v_j^*(x_h))$, $1 \leq j \leq m$, $1 \leq h \leq n$ has closed range. Note that by the definition of $\mu_{i,h}$ and the $\{w_i^{**}\}_{i=1}^n$,

$$S^{**}(w_1^{**}, w_2^{**}, \ldots, w_n^{**}) = (w_1^{**}, w_2^{**}, \ldots, w_k^{**}, 0, 0, \ldots) \in (\overbrace{X \oplus \cdots \oplus X}^{n-r+k})_\infty.$$

Hence by one of the observations above applied to the operator $\widetilde{S}$ there exist $\{w_i\}_{i=1}^n$ in X with $\sup_i \|w_i\| \le 1+\delta$ so that

$$S(w_1, w_2, \ldots, w_n) = S^{**}(w_1^{**}, w_2^{**}, \ldots, w_n^{**})$$

and $w_i^{**}(v_j^*) = v_j^*(w_i)$ for all i and j. Define now $T : E \to X$ by $Tw_i^{**} = w_i$ for $1 \le i \le r$. Note that $Tw_i^{**} = w_i = w_i^{**}$ for $1 \le i \le k$. If $r < i \le n$ then $\sum_{h=1}^{r} \mu_{i,h} w_h - w_i =$ the i'th component of $S(w_1, w_2, \ldots, w_n) =$ the i'th component of $S^{**}(w_1^{**}, w_2^{**}, \ldots, w_n^{**}) = \sum_{h=1}^{r} \mu_{i,h} w_h^{**} - w_i^{**} = 0$. Hence $Tw_i^{**} = w_i$ also for $r < i \le n$. That T is an $1+\epsilon$ isometry follows from the choice of δ and the fact that for all i,

$$1+\delta \ge \|w_i\| = \|Tw_i^{**}\| \ge \sup_j |v_j^*(w_i)| = \sup_j |w_i^{**}(v_j^*)| \ge (1+\delta)^{-1}.$$

We describe next a useful construction in Banach space theory (having roots in mathematical logic) which is related to the notion of finite representability.

Recall that a family $\mathcal{U}$ of subsets of a set I is called a *filter* if it is closed under finite intersections, does not contain the empty set, and whenever $A \subset B$ with $A \in \mathcal{U}$ then $B \in \mathcal{U}$. A maximal (with respect to inclusion) filter is called an *ultrafilter.* By Zorn's lemma every filter is contained in an ultrafilter. An ultrafilter is called *free* (or nontrivial) if the intersection of all sets in $\mathcal{U}$ is empty. An indexed family $\{x_i\}_{i\in I}$ in a topological space is said to *converge to x with respect to a filter $\mathcal{U}$* (in symbols, $x = \lim_{\mathcal{U}} x_i$) provided for every open set G containing x the set $\{i : \ x_i \in G\}$ belongs to $\mathcal{U}$. A Hausdorff space is compact if and only if every indexed family $\{x_i\}_{i\in I}$ converges (to a unique point) for every free ultrafilter $\mathcal{U}$ on I. Assume now that I is a set and $\mathcal{U}$ is a free ultrafilter on I; assume also that for all i, X_i is a Banach space. We define a seminorm $|||\cdot|||$ on $\left(\sum_i X_i\right)_\infty$ by $|||x||| = \lim_{\mathcal{U}} \|x_i\|$ where $x = \{x_i\}_{i\in I}$ with $x_i \in X_i$ for all i. The limit exists since a closed bounded interval on the line is compact. The quotient of $\left(\sum_i X_i\right)_\infty$ with respect to the closed subspace of all x with $|||x||| = 0$ with its obvious norm is a Banach space, called the *ultraproduct* of the X_i (with respect to $\mathcal{U}$), and is denoted by $\left(\prod_i X_i\right)_{\mathcal{U}}$. If all the X_i are the same space X we call the space thus obtained an *ultrapower* of X, denoted also by $X_{\mathcal{U}}$. Ultraproducts of Banach spaces are treated in detail in [9, Ch.8].

Given two families $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ of spaces and operators $T_i : X_i \to Y_i$ with $\sup_i \|T_i\| < \infty$, there is a natural operator $T : \left(\prod_i X_i\right)_{\mathcal{U}} \to \left(\prod_i Y_i\right)_{\mathcal{U}}$

called the *ultraproduct of the operators* T_i. It maps an element in $\left(\prod_i X_i\right)_{\mathcal{U}}$ represented by $x = \{x_i\}_{i\in I}$ in $(\sum X_i)_\infty$ into the element in $\left(\prod_i Y_i\right)_{\mathcal{U}}$ represented by $y = \{T_i x_i\}_{i\in I}$.

The ultraproduct of one dimensional spaces is one dimensional and more generally if $\dim X_i = n < \infty$ for all i then $\left(\prod_i X_i\right)_{\mathcal{U}}$ is also n-dimensional. On the other hand if $I = \mathbb{N}$ and $\lim_{\mathcal{U}}(\dim X_i) = \infty$ then $\left(\prod_i X_i\right)_{\mathcal{U}}$ is already nonseparable.

The ultraproduct of Banach lattices is again a Banach lattice if we take as the positive cone in $\left(\prod_i X_i\right)_{\mathcal{U}}$ the set of all elements which have representatives $x = \{x_i\}_{i\in I}$ in $\left(\sum_i X_i\right)_\infty$ with $x_i \geq 0$ for all i. If all the X_i are abstract L_p spaces for some fixed p, $1 \leq p < \infty$, then $\left(\prod_i X_i\right)_{\mathcal{U}}$ is again an abstract L_p space and hence is isometric to $L_p(\mu)$ for some measure μ by the L_p version of the Kakutani representation theorem. Similarly, if all the X_i are $C(K_i)$ spaces for some compact Hausdorff K_i then so is $\left(\prod_i X_i\right)_{\mathcal{U}}$. However, for other families of Banach spaces X_i (even e.g. if all are Orlicz spaces with the same Orlicz function Φ) the determination of the nature of $\left(\prod_i X_i\right)_{\mathcal{U}}$ is not an easy task.

As a first application of ultraproducts we shall prove now a fact mentioned already in Section 4: **If $1 \leq p \leq r \leq 2$ then $L_r(0,1)$ is isometric to a subspace of $L_p(0,1)$.** It was shown in Section 4 that for every integer n there is an isometry $T_n : \ell_r^n \to L_p(0,1)$. The ultraproduct T of the T_i is an isometry from $\left(\prod_n \ell_r^n\right)_{\mathcal{U}}$ into $L_p(0,1)_{\mathcal{U}}$ where $\mathcal{U}$ is any free ultrafilter on the positive integers. The space $\left(\prod_n \ell_r^n\right)_{\mathcal{U}}$ contains $L_r(0,1)$ as a subspace. Indeed, by using a sequence of partitions of [0,1] into n intervals (with each partition refining the previous one and so that the maximum length of the intervals tend to 0) we can represent $L_r(0,1)$ as $\overline{\bigcup_n F_n}$ where $F_n \subset F_{n+1}$ and F_n isometric to ℓ_r^n for every n. For every $g \in F_n$ let $\tilde{g} = (0,0,\ldots,0,g,g,\ldots,g\ldots) \in (\sum F_n)_\infty$ (n zeros). The closure of the images of all these $\tilde{g}$ in $\left(\prod_n F_n\right)_{\mathcal{U}}$ is isometric to $L_r(0,1)$. Thus $L_r(0,1)$ is isometric to a subspace of the abstract $L_p(\mu)$ space $L_p(0,1)_{\mathcal{U}}$. The space $L_p(\mu)$ is nonseparable but, as in any $L_p(\mu)$ space, any separable subspace of it is isometric to a subspace of $L_p(0,1)$.

Every Banach space X is isometric in a natural way to a subspace of $X_{\mathcal{U}}$;

map x into the element represented by the element $\{x_i\}_{i\in X}$ with $x_i = x$ for all i. If X is a conjugate space (and in particular a reflexive space) there is a contractive projection from $X_{\mathcal{U}}$ onto the canonical image of X in it. Map $\{x_i\}_{i\in I}$ to $w^*\text{-}\lim_{\mathcal{U}} x_i$. As for duality, the space $\left(\prod_i X_i^*\right)_{\mathcal{U}}$ can be in a natural way identified with a subspace (usually proper) of the dual of $\left(\prod_i X_i\right)_{\mathcal{U}}$.

We now discuss the relation between ultraproducts and λ-representability. It follows directly from the definitions that any ultrapower $X_{\mathcal{U}}$ of a Banach space X is finitely representable in X. There is also a converse, in a sense, to this statement. **A Banach space X is λ-representable in Y if and only if there is a subspace Z of some ultraproduct of Y so that $d(X, Z) \leq \lambda$.** The only if part is clear from the remark above. The if part is an easy generalization of the remark made above (for $L_r(0,1)$) that if X is contained in the closure of an increasing union of subspaces $\{X_n\}_{n=1}^{\infty}$ then X is isometric to a subspace of $\left(\prod_n X_n\right)_{\mathcal{U}}$ for any free ultrafilter on the integers. Indeed, let I be the set of pairs (E, ϵ) with E a finite dimensional subspace of X and $\epsilon > 0$. Introduce a partial order on I by $(E_1, \epsilon_1) < (E_2, \epsilon_2)$ if $E_1 \subset E_2$ and $\epsilon_1 > \epsilon_2$, and let $\mathcal{U}$ be an ultrafilter on I containing for all $i \in I$ the set $\{j \in I : i < j\}$. By assumption there is for every $(E, \epsilon) \in I$ an operator $T_{E,\epsilon}$ from E into Y so that $\|x\| \leq \|T_{E,\epsilon}x\| \leq (\lambda + \epsilon)\|x\|$ for all $x \in E$. For every $x \in X$ let $\tilde{x} = \{x_{E,\epsilon}\} \in \left(\sum_i Y\right)_{\infty}$ be defined by $x_{E,\epsilon} = T_{E,\epsilon}x$ if $x \in E$ and $x_{E,\epsilon} = 0$ if $x \notin E$. The image of all these $\tilde{x}$ in $Y_{\mathcal{U}}$ is easily seen to be a subspace Z which satisfies $d(X, Z) \leq \lambda$.

From the principle of local reflexivity it follows that for every Banach space X the space X^{**} **is isometric to a norm one complemented subspace of a suitable ultraproduct of X.** Without the complementation assertion this is a special case of the result above. To get the complementation assertion we have to modify the proof above. Now let I be the set of triples (E, F, ϵ), where E is a finite-dimensional subspace of X^{**}, F a finite-dimensional subspace of X^*, and $\epsilon > 0$. Introduce a partial order on I by $(E_1, F_1, \epsilon_1) < (E_2, F_2, \epsilon_2)$ if $E_1 \subset E_2$, $F_1 \subset F_2$, and $\epsilon_1 > \epsilon_2$, and let $\mathcal{U}$ be an ultrafilter on I which refines the partial order filter. For every $(E, F, \epsilon) \in I$ let $T_{E,F,\epsilon} : E \to X$ be the operator given by the principle of local reflexivity. Using these operators we define as above an isometry T from X^{**} into $X_{\mathcal{U}}$. Define a map S from $X_{\mathcal{U}}$ into X^{**} by $S(\{x_i\}) = w^*\text{-}\lim_{\mathcal{U}} x_i$. From the properties of $\{T_{E,F,\epsilon}\}$ one deduces easily that ST is the identity on X^{**}, so that TS is a projection of norm 1 from $X_{\mathcal{U}}$ onto TX^{**}.

A property (P) of Banach spaces is called a *super* property provided that if X satisfies (P) and Y is finitely representable in X, then Y satisfies (P). In particular, a super property passes from a space X to all closed subspaces of its ultraproducts. So if (P) is a hereditary property (i.e., passes to closed subspaces), a Banach space X has super (P) if and only if every ultrapower of X has (P). For example, X is superreflexive if every ultrapower of X is reflexive. An explicit local property which characterizes superreflexivity is the following: A Banach space X is superreflexive if and only if for every $\epsilon > 0$ there is an integer $N(\epsilon)$ so that any ϵ-separated dyadic tree in the unit ball of X has height $\leq N(\epsilon)$. By an ϵ-separated dyadic tree of height N we mean a set of points $\{x_{i,n} : 1 \leq i \leq 2^n,\ 1 \leq n \leq N\}$ so that $x_{i,n} = (x_{2i-1,n+1} + x_{2i,n+1})/2$ and $\|x_{2i-1,n+1} - x_{2i,n+1}\| \geq \epsilon$ for every $1 \leq i \leq 2^n$ and every $n < N$. That this condition is necessary for superreflexivity is easy to see. A reflexive space cannot contain a bounded ϵ-separated dyadic tree $\{x_{i,n}\}$ of height ∞ since the closed convex hull of such a tree is not dentable. The other direction of this implication follows from what we stated in section 6 concerning B-convexity. If X is nonreflexive it contains, for example, vectors $\{x_j\}_{j=1}^4$ of norm 1 so that for every i, $\left\|\sum_{j=1}^{i} x_j - \sum_{j=i+1}^{4} x_j\right\| \geq 3$. The set $\frac{x_1+x_2+x_3+x_4}{4}$, $\frac{x_1+x_2}{2}$, $\frac{x_3+x_4}{2}$, x_1, x_2, x_3, x_4 is a 1-separated dyadic tree of height three in the unit ball of X. In a similar manner we get in every nonreflexive space a 1-separated dyadic tree of an arbitrary finite height in the unit ball.

It is not hard to show that the existence of arbitrarily tall ϵ-separated trees in the unit ball (for some $\epsilon > 0$ or for $\epsilon = 1$) is a selfdual property and thus so is superreflexivity. Much deeper is the fact that a space is superreflexive if and only if it has an equivalent uniformly convex norm. Since uniform convexity is defined by an inequality involving four vectors the if part is obvious. The hard part is the only if part; that is, the construction of a uniformly convex equivalent norm in a space which does not have ϵ-separated dyadic trees of large height in its unit ball. The notion of dyadic trees reminds one of martingales and in fact the most elegant way to prove the only if part is to use vector-valued martingales. The proof shows in particular that if X is uniformly convex then there is an equivalent norm whose modulus of convexity $\delta(\epsilon)$ satisfies $\delta(\epsilon) \geq C\epsilon^q$ for some $C > 0$ and $q < \infty$. The proof of this fact and related material can be found in [2, 4.IV] and [6, IV.4] (see also [29]).

We introduce next a class of Banach spaces defined in terms of their finite dimensional subspaces which are closely related to $L_p(\mu)$ spaces. Let $1 \leq p \leq \infty$ and $\lambda \geq 1$. A Banach space X is called an $\mathcal{L}_{p,\lambda}$ space if for every finite dimensional subspace E of X there is a further subspace $F \supset E$ with $d(F, \ell_p^n) \leq \lambda$ where $n = \dim F$. A space is called an $\mathcal{L}_p$ space if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda < \infty$.

It is evident that every $L_p(\mu)$ space, $1 \leq p \leq \infty$, is an $\mathcal{L}_{p,1+\epsilon}$ space for every $\epsilon > 0$ (take as F a small perturbation of a subspace spanned by a finite number of suitable disjoint indicator functions). Similarly, every $C(K)$ space is an $\mathcal{L}_{\infty,1+\epsilon}$ space for every $\epsilon > 0$ (use partitions of unity). By the principle of local reflexivity, if X^{**} is an $\mathcal{L}_{p,\lambda}$ space then X is an $\mathcal{L}_{p,\lambda+\epsilon}$ space for every $\epsilon > 0$. From ultraproduct arguments used already above, it follows that: **Any $\mathcal{L}_p$ space is isomorphic to a subspace of $L_p(\mu)$ for some measure μ $(1 \le p \le \infty)$.** In particular, for $1 < p < \infty$ every $\mathcal{L}_p$ space is reflexive and any $\mathcal{L}_2$ space is isomorphic to a Hilbert space.

It is possible to make a stronger statement. Assume first that $1 < p < \infty$ and that X is an $\mathcal{L}_{p,\lambda}$ space. There is thus a set I of finite dimensional subspaces E of X, directed by inclusion, with $X = \bigcup_{E \in I} E$, so that for every E there are operators $T_E : E \to \ell_p^{n(E)}$, $S_E : \ell_p^{n(E)} \to E$ so that $\|T_E\| \le 1$, $\|S_E\| \le \lambda$ and $S_E T_E = I_E$ ($n(E) = \dim E$). Let $\mathcal{U}$ be an ultrafilter on I which refines the order filter and put $Y = \left(\prod \ell_p^{n(E)}\right)_{\mathcal{U}}$. Define T from X to Y by mapping x to the class represented by $\{T_E x\}_{E \in I}$ (in view of the choice of $\mathcal{U}$ it does not matter that $T_E x$ is defined only for E which contain x). Define $S : Y \to X$ by $S\{y_E\} = w\text{-}\lim_{\mathcal{U}} S_E y_E$ (recall that X is reflexive). Then $ST = I_X$ and Y is an $L_p(\mu)$ space. Consequently: **Every $\mathcal{L}_p$ space X, $1 < p < \infty$, is isomorphic to a complemented subspace of an $L_p(\mu)$ space.** If X is separable we deduce that X is isomorphic to a complemented subspace of $L_p(0,1)$. Similar considerations (starting with the fact proved in section 4 that ℓ_2 is isometric to a complemented subspace of $L_p(0,1)$) yield that every Hilbert space is isometric to a complemented subspace of some $L_p(\mu)$ space when $1 < p < \infty$.

In the cases $p = 1$ and $p = \infty$ we can reason similarly, but now define $\widetilde{S} : Y \to X^{**}$ by $S\{y_E\} = w^*\text{-}\lim_{\mathcal{U}} S_E y_E$ and get that $\widetilde{S}T = J_X$, the natural inclusion of X into X^{**}. By considering $T^{**} : X^{**} \to Y^{**}$ and $\widetilde{S}^{**} : Y^{**} \to X^{(iv)}$ and recalling that there is a norm one projection from $X^{(iv)}$ onto X^{**} (see section 2), it follows that if X is an $\mathcal{L}_1$ space then X^{**} is isomorphic to a complemented subspace of an $L_1(\mu)$ space. Since $L_1(\mu)^*$ is injective and since there is a norm one projection from X^{***} onto X^* we deduce that **if X is an $\mathcal{L}_1$ space then X^* is an injective Banach space.** Similarly, if X is an $\mathcal{L}_\infty$ space then the constructed ultraproduct Y is a $C(K)$ space and X^{**} is isomorphic to a complemented subspace of the injective space Y^{**}. That is, **if X is an $\mathcal{L}_\infty$ space then X^{**} is an injective Banach space.**

The converse of the previous statements essentially hold. First we show: **Let $1 < p < \infty$. A Banach space X is isomorphic to a complemented subspace of an $L_p(\mu)$ space if and only if X is an $\mathcal{L}_p$ space or X is isomorphic to a Hilbert space.** To see the "only if" direction, assume that there is a projection Q from $Y = L_p(\mu)$ onto a subspace X which is not isomorphic to a Hilbert space. By the dichotomy principle for L_p spaces, $2 < p < \infty$, discussed in section 4 (and, by duality, also for $1 < p < 2$), X has a subspace Z isomorphic to ℓ_p onto which there is a projection, say R. Let E be a finite dimensional subspace of X and $\epsilon > 0$. There is a finite dimensional subspace F of Y (a small perturbation of the span of disjoint indicator functions) containing E so that $d(F, \ell_p^m) \le (1+\epsilon)$ ($m = \dim F$) and so that there is a projection P from Y onto F with $\|P\| \le 1+\epsilon$. The space RQF is a finite dimensional subspace of Z. Hence since every element in ℓ_p has essentially a finite support it follows that there is a subspace B of Z so that $d(B, \ell_p^m) \le K$ and $\max\{\|x\|, \|y\|\} \le K\|x+y\|$ for every $x \in QF$ and $y \in B$ (K is a constant depending only on $d(Z, \ell_p)$; in fact, $K \le 1+\epsilon$ is possible).

Let $\tau : F \to B$ be an isomorphism from F onto B with $\|\tau\| \le 1$ and $\|\tau^{-1}\| \le (1+\epsilon)K$. Define $T : F \to X$ by $T = Q_{|F} + \tau(I - PQ_{|F})$. Then the restriction of T to E is the identity (since this is true of the restrictions of P and Q to E). A simple computation shows that for every $x \in F$, $(1+\epsilon)^{-1}(1+K)^{-1}K^{-1}\|x\| \le \|Tx\| \le (1+(2+\epsilon)\|Q\|)\|x\|$. Hence $G = TF$ is a subspace containing E with $d(G, \ell_p^m) \le (1+\epsilon)(1+K)K(1+(2+\epsilon)\|Q\|)$.

Consider now the case $p = 1$. Every infinite dimensional complemented subspace of an $L_1(\mu)$ contains a subspace isomorphic to ℓ_1. This is a consequence of the fact that such a space is nonreflexive (to be proved in the next section) and the structure of nonweakly compact sets in $L_1(\mu)$ discussed in section 4. Hence if X^* is injective then X^{**}, being a complemented subspace of $L_1(\mu)$ for some μ, contains a copy of ℓ_1. The proof given above for $1 < p$ carries over verbatim and yields that X^{**} and hence X is an $\mathcal{L}_1$ space.

It is also true (see [21, II.D]) that an infinite dimensional complemented subspace of a $C(K)$ space (and hence every infinite dimensional injective space) contains a copy of c_0. The same proof then yields that every space whose second dual is injective is an $\mathcal{L}_\infty$ space. Instead of using this moderately difficult structure theorem for $C(K)$ one can argue that if X^{**} is injective then X^{***} is complemented in some $L_1(\mu)$ space, hence X^{***} contains a complemented subspace isomorphic to ℓ_1, whence $X^{(iv)}$ contains a subspace isomorphic to c_0.

One consequence of the principle of local reflexivity (see [24]) is: **If X^* has the BAP then X has the BAP.** Since every $L_p(\mu)$ space, $1 \le p \le \infty$, has the BAP and the BAP passes to complemented subspaces, we conclude: **Every $\mathcal{L}_p$ space has the** BAP. A stronger statement is mentioned at the end of this section.

From the results above it follows immediately that any complemented subspace of an $\mathcal{L}_p$ space $1 \le p \le \infty$ is again an $\mathcal{L}_p$ space unless it is isomorphic to a Hilbert space (for $p = 1$ or $p = \infty$ it cannot be a Hilbert space). We also get that: **the dual of an $\mathcal{L}_p$ space is an $\mathcal{L}_{p^*}$ space.** This fact is not evident from the definition since $\mathcal{L}_p$ spaces are defined by the structure of their finite dimensional subspaces and by passing to the duals we get direct information only on the finite dimensional quotient spaces.

The $\mathcal{L}_p$ spaces give a nice local description of the complemented subspaces of

$L_p(0,1)$ for $1 < p < \infty$. From the global point of view these complemented subspaces are very hard to describe. Some global structure theorems are known (e.g. every separable $\mathcal{L}_p$ space $1 \leq p \leq \infty$ has a Schauder basis, and every separable infinite dimensional $\mathcal{L}_p$ space $1 < p < \infty$ which does not contain a copy of ℓ_2 is isomorphic to ℓ_p) but many natural questions on them are still open. For example, it is unknown whether every separable $\mathcal{L}_p$ space, $1 < p < \infty$, $(p \neq 2)$, has an unconditional basis. There are uncountably many distinct isomorphism types among the separable $\mathcal{L}_p$ spaces, $1 \leq p \leq \infty$, $(p \neq 2)$. The simplest examples for $1 < p \neq 2 < \infty$ (besides $L_p(\mu)$ spaces) are $\ell_p \oplus \ell_2$ and $(\ell_2 \oplus \ell_2 \oplus \cdots)_p$. A discussion of all of these questions is contained in [22].

Among the separable $\mathcal{L}_\infty$ spaces there are the $C(K)$ spaces with K countable compact metric which, as was pointed out in section 4, form uncountably many distinct isomorphism types. There are however spaces X such that X^* is isometric to ℓ_1 but X is not isomorphic to a $C(K)$ space. Such a space X is an $\mathcal{L}_{\infty,1+\epsilon}$ space for every $\epsilon > 0$. In fact, a space X is $\mathcal{L}_{\infty,1+\epsilon}$ for every $\epsilon > 0$ if and only if X^* is isometric to $L_1(\mu)$ for some measure μ. It is known that such an X has a subspace isometric to c_0. In contradistinction to this, there are $\mathcal{L}_{\infty,\lambda}$ spaces with $\lambda > 1$ which have the RNP (and thus in particular do not contain a subspace isomorphic to c_0). For a discussion of $\mathcal{L}_\infty$ spaces see [40].

There are a continuum of isomorphism classes of separable $\mathcal{L}_1$ spaces. The simplest example (besides ℓ_1 and $L_1(0,1)$) is the kernel of a quotient map $T : \ell_1 \to L_1$. The isomorphism type of this kernel turns out not to depend on the choice of T. This kernel is an $\mathcal{L}_1$ space which is not isomorphic to a complemented subspace of an $L_1(\mu)$ space. For a discussion of $\mathcal{L}_1$ spaces, see ([22]).

There is also a "local view" of Banach lattices. A Banach space X is said to have *Gordon-Lewis local unconditional structure (GL-l.u.st.)* if there is a constant λ so that for every finite dimensional subspace E of X there is a space Y with an unconditional basis and operators: $T : E \to Y$, $S : Y \to X$ so that $ST = I_E$ and $\|T\|\ \|S\| uc(Y) \leq \lambda$ where $uc(Y)$ denotes the unconditional constant of a basis in Y. Note that by a simple perturbation argument we can assume that Y is always finite dimensional (we may have to replace λ by any $\lambda' > \lambda$). If always Y can be chosen to be a subspace of X which contains E, X is said to have *local unconditional structure (l.u.st.)*. Then of course T and S can be taken to be the inclusion maps. It is evident that l.u.st. implies GL-l.u.st., but whether the converse is true is open. Later in this section we mention a partial converse.

Any Banach lattice X has l.u.st. where the λ in the definition can be taken as any constant > 1. Indeed, if X is an order complete lattice, then we saw in section 5 that for any finite dimensional subspace E and every $\epsilon > 0$ there are vectors $\{x_1, \ldots, x_m\}$ in X whose span contains E and which is an ϵ-perturbation of a sequence of disjointly supported vectors. Since a disjointly supported sequence in a lattice is 1-unconditional and the bidual of a Banach lattice is an order complete lattice, the general case follows from the principle of small perturbations and the principle of local reflexivity.

It is clear from the definition that $\mathcal{L}_p$ spaces, $1 \le p \le \infty$, have l.u.st.

The close relationship between l.u.st. and lattice structure is expressed in the following result: **A Banach space X has GL-l.u.st. if and only if X^{**} is isomorphic to a complemented subspace of a Banach lattice.** Indeed, since by the definition a complemented subspace of a space with GL-l.u.st. has GL-l.u.st., it follows that if X^{**} is a complemented subspace of a lattice it has GL-l.u.st. By the principle of local reflexivity we deduce that also X has GL-l.u.st.

To prove the other direction we use ultraproducts. Assume X has GL-l.u.st. with some constant λ. For every finite dimensional subspace E of X there are a space Y_E with 1-unconditional basis and operators $T_E : E \to Y_E$ and $S_E : Y_E \to X$ with $\|T_E\| \le 1$, $\|S_E\| \le \lambda$ and $S_E T_E = I_E$. Let I be the directed set of all finite dimensional subspaces of X and let $\mathcal{U}$ be an ultrafilter on I which refines the order filter. Let $T : X \to Y = (\prod Y_E)_{\mathcal{U}}$ be defined by $Tx = \{T_E x\}$ (as before, because of our assumption on $\mathcal{U}$ it does not matter that $T_E x$ is defined just for the E containing x). Let $S : Y \to X^{**}$ be defined by $S\{y_E\} = w^* - \lim S_E y_E$. The space Y is a Banach lattice and $ST = I_X$. Hence T^{**} is an isomorphism from X^{**} into the lattice Y^{**} and PS^{**} is a projection from Y^{**} onto $T^{**}(X^{**})$ where P is a projection from $X^{(iv)}$ onto X^{**} (see section 2).

An immediate consequence of the claim just proved is that X **has GL-l.u.st. if and only if X^* has GL-l.u.st.** Indeed, if X has GL-l.u.st. then X^{**} is complemented in a Banach lattice. Hence the same is true for X^{***} and thus X^* has GL-l.u.st. Conversely, if X^* has GL-l.u.st. then X^{***}, hence also $X^{(iv)}$, is complemented in a Banach lattice. Since X^{**} is complemented in $X^{(iv)}$, it follows that also X has GL-l.u.st.

The argument given earlier that if a complemented subspace X of an $L_p(\mu)$ space is not isomorphic to a Hilbert space then it is an $\mathcal{L}_p$ space depended on the fact that an $L_p(\mu)$ space is finitely representable in ℓ_p and that ℓ_p is, for some λ, λ-finitely representable in every finite codimensional subspace of X. The same argument shows that if Y is complemented in a Banach lattice X and there is $\lambda < \infty$ so that X is λ-finitely representable in every finite codimensional subspace of Y, then Y has l.u.st. From this and the characterization of GL-l.u.st. mentioned above we deduce: X **has GL-l.u.st. if and only if $X \oplus c_0$ has l.u.st.**

Among the spaces which fail to have GL-l.u.st. important examples are the spaces of compact and bounded operators on ℓ_2. In fact, the only unitary ideal of operators on ℓ_2 (a notion discussed in section 10) which has GL-l.u.st. is the space of Hilbert-Schmidt operators, which is itself a Hilbert space. Spaces of analytic functions, such as the disc algebra, typically do not have GL-l.u.st. For a proof of most of these statements see [9, Ch.17], [20], or [27].

We conclude this section by mentioning briefly a quantitative notion of the approximation property and its relation to ultrapowers. A Banach space X is said to have the *uniform approximation property* (UAP) if there is a constant λ and a *uniformity function* $\varphi(\cdot)$ on the natural numbers so that for every finite dimensional subspace E of X there is $T \in B(X, X)$ so that $\|T\| \leq \lambda$, $T_{|E} = I_E$, and $\dim TX \leq \varphi(\dim E)$. **A Banach space X has the UAP if and only if every ultrapower of X has the BAP.** It follows from this that every $\mathcal{L}_p$ space, $1 \leq p \leq \infty$, has the UAP. It also follows, in view of results about ultrapowers presented earlier, that if X^{**} fails the BAP then X fails the UAP. From this it can be deduced (see [24]) that there are Banach spaces which have the BAP but not the UAP (in fact, $K(\ell_2, \ell_2)$ is such a space).

While it is usually easy to check whether a concrete separable space has the BAP, it is often quite difficult to decide whether a special space has the UAP. For example, it is open whether the disk algebra has the UAP. As a matter of fact, the only general class of spaces other than L_p spaces which are known to have the UAP are the reflexive Orlicz spaces.

10 Some special classes of operators

In this section we shall discuss some special classes of linear operators between Banach spaces and their relation to the geometry of Banach spaces. We shall also discuss some other topics in operator theory which are relevant to the study of the structure of Banach spaces.

Most of the classes of operators we consider have the *ideal property*, meaning that the operators from a fixed X into a fixed Y having this property form a linear space and that whenever $T : X \to Y$ belongs to this class then for every bounded $U : Z \to X$ and $V : Y \to W$ the operator VTU belongs to this class. The three most elementary operator ideals are the bounded operators, the compact operators, and the weakly compact operators. Since $B(X, Y)$, $K(X, Y)$, and $WK(X, Y)$ are all Banach spaces under the operator norm, the operator norm is the natural norm to use for operators in these classes. A Banach space X has the *Dunford-Pettis (DP) property* provided that every weakly compact operator with domain X maps weakly compact sets into norm compact sets. It is clear that if $T \in WK(X, Y)$, $S \in WK(Y, Z)$, and Y has the DP property, then ST is a compact operator. In particular, if X has the DP property and P is a weakly compact projection on X, then $P = P^2$ is compact. This means that the only complemented reflexive subspaces of X are the finite dimensional ones.

A subset of a Banach space is relatively weakly compact if and only if it is relatively weakly sequentially compact. Therefore, a space X has the DP property if and only if every weakly compact operator with domain X maps sequences which converge weakly to zero into sequences which converge in norm to zero.

The following is an elegant characterization of spaces having the DP property. **X has the DP property if and only if whenever $\{x_n\}_{n=1}^\infty$ in X tends weakly to 0 and $\{x_n^*\}_{n=1}^\infty$ in X^* tends weakly to 0 the sequence of scalars $\{x_n^*(x_n)\}_{n=1}^\infty$ tends to** 0. Indeed, assume that X has DP and $x_n^* \xrightarrow{w} 0$. The operator $T : X \to c_0$ defined by $Tx = (x_1^*(x), x_2^*(x), \ldots)$ is a weakly compact operator (it is easier to check that T^* is weakly compact). Hence, if $x_n \to 0$ weakly, $|x_n^*(x_n)| \le \|Tx_n\| \to 0$. Conversely, assume the condition on the sequences is satisfied, $T : X \to Y$ is weakly compact, and $x_n \xrightarrow{w} 0$. If $\|Tx_n\| \not\to 0$ we may assume, by passing to a subsequence, that $\|Tx_n\| \ge \delta$ for some $\delta > 0$ and all n. Let $y_n^* \in X^*$ be such that $y_n^*(Tx_n) = \|Tx_n\|$ and $\|y_n^*\| = 1$ for all n. Since T^* is weakly compact as well we may assume (passing again to a subsequence if needed) that $T^*y_n^* \xrightarrow{w} x^*$ for some x^*. Then

$$0 = \lim_n (T^*y_n^* - x^*)(x_n) = \lim_n y_n^*(Tx_n) = \lim_n \|Tx_n\|$$

a contradiction.

It follows from the criterion we just proved that X has the DP property if X^* has this property. (The converse is false: $\left(\sum_{n=1}^\infty \ell_2^n\right)_1$ has the DP property but $\left(\sum_{n=1}^\infty \ell_2^n\right)_\infty$ contains a complemented copy of ℓ_2 and hence fails the DP property.) It is also clear that a complemented subspace of a space with the DP property has the same property. We shall verify shortly that $C(K)$ spaces have the DP property. Consequently we get that: **All $\mathcal{L}_\infty$ and $\mathcal{L}_1$ spaces have the DP property**.

To show that $C(K)$ has the DP property let $\{x_n\}_{n=1}^\infty$ be a sequence in the unit ball of $C(K)$ so that $x_n(t) \to 0$ for every $t \in K$. Let $\{\mu_n\}_{n=1}^\infty$ be a sequence in the unit ball of $C(K)^*$ which tends weakly to 0. All the μ_n can be considered as elements in $L_1(\mu)$ where $\mu = \sum_{n=1}^\infty |\mu_n|/2^n$. By the analysis done in section 4 of sets in $L_1(\mu)$ which have weakly compact closure it follows that for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever E is a Borel set in K with $\mu(E) < \delta$ then $|\mu_n|(E) < \epsilon$ for all n. By Egoroff's theorem there is a set E with $\mu(E) < \delta$

so that $|x_n(t)| \leq \epsilon$ for all t outside E and $n \geq n(\epsilon)$. Hence for such n,

$$\left| \int_K x_n d\mu_n \right| \leq \left| \int_E x_n d\mu_n \right| + \left| \int_{K\sim E} x_n d\mu_n \right| \leq \epsilon + \epsilon;$$

that is, $\int_K x_n d\mu_n \to 0$.

We also want to single out the strictly singular operators. An operator $T : X \to Y$ is *strictly singular* if it is not an isomorphism on any infinite dimensional subspace of X. Of course, any compact operator is strictly singular but the converse is false. The formal identity map from ℓ_p into ℓ_r where $p < r$ is a simple example of a strictly singular operator which is not compact. Note that, by the discussion in section 3, an operator from ℓ_p, $1 \leq p < \infty$, into itself (or from c_0 into itself) is strictly singular if and only if it is compact. A more sophisticated and interesting example of a strictly singular operator is the formal identity operator $I_p : C(K) \to L_p(\mu)$, $1 \leq p < \infty$, where μ is a finite measure on the compact set K. The operator I_p is not compact if, for example, $K = [0, 1]$ and μ is Lebesgue measure (in fact, I_p is compact only if μ is purely atomic). Observe that if X is a subspace of $C(K)$, the statement that the restriction of I_p to X is an isomorphism means that the supremum norm and the $L_p(\mu)$ norm are equivalent on X. If this occurs for *some* $p < \infty$, then the extrapolation argument used in section 4 shows that the supremum norm and the $L_r(\mu)$ norm are equivalent on X for *all* $1 \leq r < \infty$. Thus it is enough to check e.g. that I_1 or I_2 is strictly singular. That I_2 is strictly singular is an exercise in textbooks (e.g. [18, Ch. 10 # 41, # 55]), but we prove it later in this section when we discuss p-summing operators.

Unlike the situation with compact or weakly compact operators, the dual of a strictly singular operator need not be strictly singular. If T is a quotient map from ℓ_1 onto a separable space X not containing a subspace isomorphic to ℓ_1 (e.g. $X = \ell_2$) then T is strictly singular but T^* is an isometric embedding of X^* into ℓ_∞.

Although strictly singular operators need not be compact they have compact restrictions to infinite dimensional subspaces. Indeed, by the same method used in section 3 to prove the existence of a basic sequence in any Banach space one can prove the following. If $T : X \to Y$ is not an isomorphism on any subspace of X of finite codimension then there is for every $\epsilon > 0$ a normalized basic sequence $\{x_n\}_{n=1}^\infty$ in X (with basis constant two, say) so that $\|Tx_n\| \leq 4^{-n}\epsilon$ for every $n \geq 1$. The restriction of T to the span of $\{x_n\}_{n=1}^\infty$ is a compact operator of norm $\leq \epsilon$.

It follows from this observation that the sum of two strictly singular operators is strictly singular and thus the class of strictly singular operators has the ideal property. It is also easy to check that $SS(X,Y)$ is complete in the operator norm.

Two Banach spaces X and Y are called *totally incomparable* if there is no infinite dimensional space Z which is isomorphic to subspaces of both X and Y. **If every operator from X to Y is strictly singular** (and in particular if X and Y are totally incomparable) **and Z is a complemented subspace of $X \oplus Y$ then there is an automorphism T of $X \oplus Y$ so that TZ is of the form $TZ = X_0 \oplus Y_0$ where X_0 (respectively Y_0) is a complemented subspace of X (respectively Y)**. This result is not as simple as the preceding results in this section. Its proof can be found in [11] 2.c.13.

Another class of operators is that of *Fredholm operators*. An operator $T : X \to Y$ is called a Fredholm operator if $\alpha(T) := \dim \ker T < \infty$ and TX is closed with $\beta(T) := \dim Y/TX < \infty$. The number $i(T) = \alpha(T) - \beta(T)$ is called *the index of* T and has a significant role in many applications. As a matter of fact $i(T)$ can be defined also for non-Fredholm operators provided TX is closed and at least one of the numbers $\alpha(T)$ or $\beta(T)$ is finite (in this case the index can be of course either $+\infty$ or $-\infty$). The class of Fredholm operators is not closed under addition or composition. However if T_1 and T_2 are Fredholm operators so is T_1T_2 (provided it is properly defined) and $i(T_1T_2) = i(T_1) + i(T_2)$. Also $i(T^*) = -i(T)$. These facts are simple exercises in linear algebra. Somewhat harder to prove is that $Fr(X,Y)$ is preserved by strictly singular perturbations. **If $T, S : X \to Y$ with T Fredholm and S strictly singular then $T + S$ is also Fredholm with** $i(T+S) = i(T)$ (see [11] 2.c.9).

An operator $T : X \to Y$ is called *absolutely summing* if whenever $\sum_{i=1}^{\infty} x_i$ converges unconditionally in X the series $\sum_{i=1}^{\infty} Tx_i$ converges absolutely. From the closed graph theorem one deduces easily that T is absolutely summing if and only if there is a constant C so that for any choice of $\{x_i\}_{i=1}^n$ in X

$$\sum_{i=1}^{n} \|Tx_i\| \le C \sup\left\{\sum_{i=1}^{n} |x^*(x_i)| : \|x^*\| \le 1\right\}.$$

More generally, for every $0 < p < \infty$ we can define the class of *p-summing operators* as those operators for which there is a constant C so that for all choices of $\{x_i\}_{i=1}^n$ in X

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \le C \sup\left\{\left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{1/p} : \|x^*\| \le 1\right\}. \tag{28}$$

Thus 1-summing is the same as absolutely summing. The smallest constant

C for which (28) holds is denoted by $\pi_p(T)$. If T is not p-summing we put $\pi_p(T) = \infty$. The set of all $T : X \to Y$ with $\pi_p(T) < \infty$ is denoted by $\Pi_p(X,Y)$. It is trivial to check that for $1 \le p < \infty$, $\Pi_p(X,Y)$ is a Banach space with $\pi_p(T)$ as the norm. It is also clear from the definition that whenever ST is defined and one of the operators S, T is p-summing so is their product and $\pi_p(ST) \le \|S\|\pi_p(T)$ and $\pi_p(ST) \le \pi_p(S)\|T\|$. In particular the class of p summing operators has the ideal property. Sometimes one uses also the notation $\pi_\infty(T)$. It naturally means just the usual operator norm $\|T\|$ of T.

Without further mention we shall henceforth assume that $p \ge 1$ when discussing p-summing operators. However, there are important applications to Banach space theory of p-summing operators for $0 < p < 1$ (see, e.g., the article [32]), and much of the theory of p-summing operators, $1 \le p < \infty$, goes over to the range $0 < p < 1$. In particular, there is a version of the Pietsch factorization theorem (discussed below when $1 \le p < \infty$) for p-summing operators, $0 < p < 1$ (see [20, Th. 9.1]).

Notice that the supremum on the right side of (28) is the norm of the operator from $\ell_{p^*}^n$ to X defined by $e_k \mapsto x_k$ (notice that $p \ge 1$ is needed for this). Thus $\pi_p(T)$ can also be defined by

$$\pi_p(T) = \sup \left\{ \left(\sum_{n=1}^{\infty} \|TVe_n\|^p \right)^{1/p} : V : \ell_{p^*} \to X,\ \|V\| \le 1 \right\}. \tag{29}$$

Equation (29) implies that $\pi_p(T)$ is the supremum of $\pi_p(TV)$ as V varies over the norm one operators from ℓ_{p^*} into X.

In contrast to the classes of weakly compact or strictly singular operators, the class of p-summing operators is determined by the behavior of T on the finite dimensional subspaces of X. Because of the quantitative nature of its definition the notion of p-summing norm plays also an important role in the study of finite dimensional spaces.

If K is a compact Hausdorff space and μ a regular probability measure on K then the identity operator I_p from $C(K)$ into $L_p(\mu)$ is easily seen to be a p-summing operator with p-summing norm one. It turns out that this simple example is the prototype of general p-summing operators. This is the content of the *Pietsch factorization theorem.* **Let X be a subspace of $C(K)$. An operator $T : X \to Y$ is p-summing, $1 \le p < \infty$, if and only if there is a regular probability measure μ on K and a constant C so that for all x in X,**

$$||Tx\| \le C \left(\int_K |x(t)|^p d\mu(t) \right)^{1/p}. \tag{30}$$

Moreover, the smallest C for which (30) holds is $\pi_p(T)$. Note also that the "only if" part can always be applied with $K = B_{X^*}$ under the weak* topology, or with K the weak* closure of the extreme points of B_{X^*}, again in the weak* topology.

In diagrammatic form, the Pietsch factorization can be stated as follows. Let X be a subspace of $C(K)$. If T is p-summing then there is a regular probability measure μ on K and an operator S from $X_p := \overline{I_p X}$ into Y so that $T = SI_{p|X}$; moreover, $\|S\| = \pi_p(T)$.

$$\begin{array}{ccc}
C(K) & \xrightarrow{I_p} & L_p(\mu) \\
\cup & & \cup \\
X & \xrightarrow{I_{p|X}} & X_p \\
 & \overset{T}{\searrow} & \downarrow S \\
 & & Y
\end{array}$$

(For $p = 2$, the operator S can be extended to an operator from $L_p(\mu)$ into Y, but for $p \neq 2$ this is not in general the case.)

To prove the Pietsch factorization theorem, assume that $\pi_p(T) = 1$. Consider the following subsets of $C(K)$

$$\begin{aligned}
G &= \{f \in C(K) : \sup\{f(t) : t \in K\} < 1\} \\
F &= \operatorname{conv}\{f \in C(K) : f(\cdot) = |x(\cdot)|^p, \|Tx\| = 1\}.
\end{aligned}$$

The sets F and G are convex with G open and, since $\pi_p(T) = 1$, $F \cap G = \emptyset$. By the separation theorem there is a regular signed measure μ on K and a positive λ so that $\int f \; d\mu < \lambda$ for $f \in G$ and $\int f \; d\mu \geq \lambda$ for $f \in F$. Since whenever $g \leq f$ and $f \in G$ also $g \in G$ it follows that the measure μ has to be positive so after normalization we may assume that it is a probability measure. Also since G contains the open unit ball of $C(K)$ we must have $\lambda \geq 1$. Hence for every $x \in X$, $\|Tx\|^p \leq \int_K |x(t)|^p d\mu(t)$.

From this factorization theorem we can easily deduce several interesting facts. First, whenever $1 \leq p \leq r < \infty$, $\pi_r(T) \leq \pi_p(T)$ and thus $\Pi_p(X,Y) \subset \Pi_r(X,Y)$. (This is trivially true also for $r = \infty$.) Next, since $L_p(\mu)$ is reflexive for $1 < p < \infty$, every operator in $\Pi_p(X,Y)$ is weakly compact. By the preceding remark this is true also for $p = 1$. A third consequence is that the identity operator on an infinite dimensional space X is never p-summing for any $p < \infty$. Indeed, by the Pietsch factorization theorem, this is equivalent to the assertion that $I_p : C(K) \to L_p(\mu)$ is strictly singular for any probability measure μ on any compact Hausdorff space K. As we mentioned earlier, if I_p were not strictly singular for some $p < \infty$, then there would exist an infinite dimensional subspace X of $C(K)$ so that for all $1 \le p < \infty$, the $L_p(\mu)$ norm is equivalent to the supremum norm. So X would be isomorphic to an infinite dimensional Hilbert space and the identity operator on X (hence also the identity operator on ℓ_2) would be 1-summing. But obviously there exists in ℓ_2 an unconditionally converging series which is not absolutely summable.

Recall that we already proved in section 8 that the identity operator on an infinite dimensional space is not absolutely summing (in fact, a stronger statement was proved). The two proofs we have given for this result are very different but have in common that Hilbert spaces enter naturally into them.

There are many other places where the Pietsch factorization theorem is useful, in particular, for proving the *composition inequality* for p-summing operators, which says **if** $T \in \Pi_p(X,Y)$ **and** $S \in \Pi_q(Y,Z)$**, then** $ST \in \Pi_r(X,Z)$ **and** $\pi_r(ST) \le \pi_p(T)\pi_q(S)$**, where** $1/r := 1 \wedge (1/p + 1/q)$**.** See ([9, 2.22]). The Pietsch factorization theorem also makes it easy to check that $\pi_p(T^{**}) \le \pi_p(T)$ for any operator T (see [9, 2.19]). This fact can also be proved using the principle of local reflexivity.

Next we examine ideals of operators on ℓ_2 and how they are related to p-summing operators. If T is a compact operator on a Banach space then we can define a (finite or infinite) sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$ consisting of the nonzero eigenvalues of T, repeated according to multiplicity, and ordered so that $|\lambda_1(T)| \ge |\lambda_2(T)| \ge \cdots$. If the sequence is infinite, then necessarily $\lambda_n(T) \to 0$. For the Banach space ℓ_2, the spectral theorem for compact operators says that if $T \in K(H,H)$, then there is an orthonormal basis $\{x_n\}_{n=1}^{\infty}$ so that the self-adjoint operator T^*T is represented as

$$T^*Tx = \sum_{n \in P} \lambda_n(T^*T)\langle x, x_n\rangle x_n \tag{31}$$

with $\lambda_n(T^*T) > 0$ for $n \in P$ and $T^*Tx_n = 0$ for $n \notin P$.

Now if X is a Banach space with a 1-symmetric basis $S = \{e_n\}_{n=1}^{\infty}$, let $S(\ell_2)$ be those compact operators on ℓ_2 for which $\sum \sqrt{\lambda_n(T^*T)}\, e_n$ converges in X, and, for $T \in S(\ell_2)$, set

$$\sigma_S(T) := \left\| \sum \sqrt{\lambda_n(T^*T)} e_n \right\|_X . \tag{32}$$

Then $(S(\ell_2), \sigma_S)$ is a Banach space (verifying the triangle inequality is a bit tricky). It is less difficult to check that $S(\ell_2)$ satisfies the *unitary ideal property*

$$\sigma_S(UTV) \le \|U\|\sigma_S(T)\|V\|$$

for all $U, V \in B(H,H)$ and $T \in S(\ell_2)$.

When S is the unit vector basis for $\ell_p, 1 \le p < \infty$, the resulting ideal, called the *Schatten-von Neumann p-class*, is denoted by (S_p, σ_p). The ideal (S_2, σ_2), which is a Hilbert space, is called the *Hilbert-Schmidt class*. The Schatten-von Neumann classes are discussed in many books, including [9] and [21].

Given the representation (31) for T^*T with $T \in K(H,H)$, it is evident that $\{Tx_n\}_{n=1}^\infty$ is orthogonal and $\|Tx_n\|^2 = \lambda_n(T^*T)$, and hence $\sigma_p(T) = \left(\sum_{n=1}^\infty \|Tx_n\|^p\right)^{1/p}$ for $1 \le p < \infty$. In the Hilbert-Schmidt case, $\sigma_2(T) = \left(\sum_{n=1}^\infty \|Ty_n\|^2\right)^{1/2}$ for *any* orthonormal basis $\{y_n\}_{n=1}^\infty$ (just write $\|Ty_m\|^2 = \langle T^*Ty_m, y_m\rangle$ and expand y_m with respect to $\{x_n\}_{n=1}^\infty$). From this and the ideal property of S_2 it follows easily that $\Pi_2(\ell_2, \ell_2) = S_2$ with equality of the respective norms. Indeed, if $T \in K(\ell_2)$ and $\{x_n\}_{n=1}^\infty$ is an orthonormal basis, then by (29) $\sigma_2(T)^2 = \sum\limits_{n=1}^\infty \|Tx_n\|^2 \le \pi_2(T)^2$. On the other hand, if $V : \ell_2 \to \ell_2$ and $\|V\| \le 1$, then $\left(\sum_{n=1}^\infty \|TVx_n\|^2\right)^{1/2} \le \sigma_2(TV) \le \sigma_2(T)$, so (29) gives $\pi_2(T) \le \sigma_2(T)$.

For an operator $T : \ell_2 \to X$, it is not true that $\pi_2(T)^2 = \sum \|Tx_n\|^2$ for every orthonormal basis $\{x_n\}_{n=1}^\infty$, but it is not very hard to show that $\pi_2(T)^2$ is the supremum over all orthonormal bases of $\sum \|Tx_n\|^2$. More importantly, there is a finite version [20, 18.4] of this which is useful for studying finite dimensional spaces. This lemma says: **If T is an operator from ℓ_2^n into some Banach space, then there is an orthonormal basis $\{x_i\}_{i=1}^n$ so that $\pi_2(T)^2 \le 2\sum\limits_{i=1}^n \|Tx_i\|^2$.**

We next discuss p-summing operators on ℓ_2 for other values of p: **For every $1 \le p < \infty$, the p-summing operators on ℓ_2 coincide with the Hilbert-Schmidt operators**, and of course $\pi_p(\cdot)$ is equivalent to $\sigma_2(\cdot)$. First a preliminary remark. Given $T \in K(H,H)$ with T^*T represented as in (31), define $|T|x = \sum \sqrt{\lambda_n(T^*T)}\langle x, x_n\rangle x_n$. There is an obvious partial isometry (i.e., an orthogonal projection followed by an isometry) U so that $T = U|T|$. This is the *polar decomposition* of the compact operator T. Suppose now that T is Hilbert-Schmidt. We want to see that T is 1-summing (and hence p-summing for every $1 \le p < \infty$). By considering the polar decomposition of T, we can assume that there is an orthonormal basis $\{e_n\}_{n=1}^\infty$ and $\lambda_n \ge 0$ so that $Te_n = \lambda_n e_n$ for every n and $1 = \sigma_2(T) = \sum\limits_{n=1}^\infty \lambda_n^2$. Let $\{\epsilon_n\}_{n=1}^\infty$ be a Rademacher sequence. Using Khintchine's inequality (1) we get

$$\sup\left\{\sum_{i=1}^n |\langle x_i, x\rangle| : \|x\| \le 1\right\} \ge \sup_{\pm}\left\{\sum_{i=1}^n \left|\left\langle x_i, \sum_{j=1}^\infty \pm\lambda_j e_j\right\rangle\right|\right\}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{n} \mathbb{E} \left| \sum_{j=1}^{\infty} \lambda_j \epsilon_j \langle x_i, e_j \rangle \right| \\
&\geq A_1 \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} |\lambda_j \langle x_i, e_j \rangle|^2 \right)^{1/2} = A_1 \sum_{i=1}^{n} \|T x_i\|.
\end{aligned}
$$

This shows that $\pi_1(T) \leq A_1^{-1} \sigma_2(T)$.

To show the converse, it is enough to check that if $T \in \Pi_p(\ell_2, \ell_2)$ with $2 \leq p < \infty$ then T is 2-summing. More generally: **If $T \in \Pi_p(X, Y)$ with $2 \leq p < \infty$ and Y has cotype 2 then $T \in \Pi_2(X, Y)$**. Since T is p-summing, the Pietsch factorization theorem kindly supplies a probability measure μ on B_{X^*} so that for all x in X,

$$
\|Tx\| \leq \pi_p(T) \left(\int_{B_{X^*}} |x^*(x)|^p d\mu(x^*) \right)^{1/p}.
$$

Using Khintchine's inequality (1) in the last step, we get

$$
\begin{aligned}
\left(\sum_{i=1}^{n} \|T x_i\|^2 \right)^{1/2} &\leq C_2(Y) \left(\mathbb{E} \left\| \sum_{i=1}^{n} \epsilon_i T x_i \right\|^2 \right)^{1/2} \\
&\leq C_2(Y) \left(\mathbb{E} \left\| T \left(\sum_{i=1}^{n} \epsilon_i x_i \right) \right\|^p \right)^{1/p} \\
&\leq C_2(Y) \pi_p(T) \left(\mathbb{E} \int_{B_{X^*}} \left| x^* \left(\sum_{i=1}^{n} \epsilon_i x_i \right) \right|^p d\mu(x^*) \right)^{1/p} \\
&\leq C_2(Y) \pi_p(T) B_p \sup \left\{ \left(\sum_{i=1}^{n} |x^*(x_i)|^2 \right)^{1/2} : \|x^*\| \leq 1 \right\},
\end{aligned}
$$

so that $\pi_2(T) \leq B_p C_2(Y) \pi_p(T)$.

We prove next an important inequality, called *Grothendieck's inequality*, which is an essential tool in studying p-summing operators as well as other topics. The inequality states: **There is a universal constant K_G so that whenever $A = (a_{ij})_{i,j=1}^n$ is a scalar matrix so that $\left| \sum_{i,j=1}^{n} a_{ij} t_i s_j \right| \leq 1$ for all $\{t_i\}_{i=1}^n$ and $\{s_j\}_{j=1}^n$ of absolute value ≤ 1 then $\left| \sum_{i,j=1}^{n} a_{ij} \langle x_i, y_j \rangle \right| \leq K_G$ for all choices of vectors $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ in the unit ball of a Hilbert space H**. A more conceptual (but obviously equivalent) way of stating Grothendieck's inequality is the following. Suppose that the matrix $A = (a_{ij})_{i,j=1}^n$ has norm at most one when considered as an operator from ℓ_∞^n to ℓ_1^n. Let H be a Hilbert space. Then $\|A \otimes I_H\| \le K_G$ as an operator from $\ell_\infty^n(H)$ to $\ell_1^n(H)$, where

$$(A \otimes I_H)(y_1, \ldots, y_n) := \left(\sum_{j=1}^n a_{1j} y_j, \ldots, \sum_{j=1}^n a_{nj} y_j \right)$$

for $(y_1, \ldots, y_n)$ in $\ell_\infty^n(H)$.

To prove Grothendieck's inequality fix n and a matrix $A = (a_{ij})_{i,j=1}^n$ with $\|A\| := \|A : \ell_\infty^n \to \ell_1^n\| = 1$ as above, and put $!A! := \|A \otimes I_H : \ell_\infty^n(H) \to \ell_1^n(H)\|$. We have to find an estimate on $!A!$ independent of n and A. Note first that for all u_i and v_j in a Hilbert space

$$\left| \sum_{i,j=1}^n a_{ij} \langle u_i, v_j \rangle \right| \le !A! \max_i \|u_i\| \max_j \|v_j\|.$$

Since all Hilbert spaces are created equal, we can use for H any (infinite dimensional or even $2n$-dimensional) Hilbert space. For the purpose of proving Grothendieck's inequality, some Hilbert spaces are more equal than others! We use for H the subspace of $L_2(0,1)$ mentioned in section 4 consisting just of functions having a Gaussian distribution with mean 0. To prove Grothendieck's inequality it is enough to consider norm one vectors $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ in H. Given $0 < \delta < 1/2$ there is an $M = M(\delta)$ so that for any norm one function f in H, $\|f - f^M\|_2 = \delta$ where

$$f^M(t) = \begin{cases} f(t) & \text{if } |f(t)| \le M \\ M \text{ sign } f(t) & \text{if } |f(t)| > M. \end{cases}$$

Note that by our assumption on the matrix A, for any choice of functions f_i and g_j in $L_2(0,1)$ which are uniformly bounded by M,

$$\left| \sum_{i,j} a_{ij} \langle f_i, g_j \rangle \right| = \int_0^1 \left| \sum_{i,j} a_{ij} f_i(t) \overline{g_j(t)} \right| dt \le M^2.$$

Hence

$$\left| \sum_{i,j} a_{ij} \langle x_i, y_j \rangle \right| \le \left| \sum_{i,j} a_{i,j} \langle x_i^M, y_j^M \rangle \right| + \left| \sum_{i,j} a_{ij} \langle x_i, y_j - y_j^M \rangle \right| + \\ + \left| \sum_{i,j} a_{ij} \langle x_i - x_i^M, y_j^M \rangle \right|$$

$$\le M^2 + 2!A!\delta.$$

Since the supremum of the left hand side (over all choices of x_i and y_j) is by definition $!A!$, we deduce that $!A! \le \frac{M^2}{1-2\delta}$. This completes the shortest proof we know for Grothendieck's inequality. With the optimal choice of δ this proof yields that $K_G < 8.69$ (or $K_G < 8.55$ if one exercises more care in the computation), but it is known that $K_G < 1.79$. Incidentally, although the best value for K_G is unknown, it is known that for complex scalars it is smaller than for real ones.

The first application of Grothendieck's inequality is: **Every bounded linear operator T from ℓ_1 to ℓ_2 is absolutely summing and $\pi_1(T) \le K_G\|T\|$.** Indeed, let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis for ℓ_1 and let $u_i = \sum\limits_{j=1}^{m} a_{ij}e_j$ be n vectors in ℓ_1^m for some m so that $\sum\limits_{i=1}^{n} |x^*(u_i)| \le 1$ for every unit vector x^* in ℓ_1^*. For any choice $\{s_j\}_{j=1}^{m}$ of scalars of absolute value ≤ 1 let $x^* \in \ell_\infty^m$ be defined by $x^*(e_j) = s_j$. Then for all $\{t_i\}_{i=1}^{n}$ with $|t_i| \le 1$,

$$\left|\sum_j \sum_i a_{ij} t_i s_j\right| \le \sum_i |t_i| \left|\sum_j a_{ij} x^*(e_j)\right| \le \sum_i |x^*(u_i)| \le 1.$$

Hence, if $y_i \in \ell_2$ with $\langle Tu_i, y_i\rangle = \|Tu_i\|$ and $\|y_i\| = 1$ for all i, we get

$$\sum_i \|Tu_i\| = \sum_i \langle Tu_i, y_i\rangle = \sum_j \sum_i a_{ij}\langle Te_j, y_i\rangle \le K_G\|T\|.$$

By the local nature of the definition of the 1-summing norm it follows immediately that if X is an $\mathcal{L}_{1,\lambda}$ space and Y an $\mathcal{L}_{2,\mu}$ space then for every $T : X \to Y$, $\pi_1(T) \le \lambda\mu K_G\|T\|$.

As a second application of Grothendieck's inequality, we prove a result alluded to in section 5: **If T is an operator from a Banach lattice X into a Banach lattice Y, then for all $\{x_i\}_{i=1}^{n} \subset X$**

$$\left\|\left(\sum_{i=1}^{n} |Tx_i|^2\right)^{1/2}\right\| \le K_G\|T\| \left\|\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right\|. \tag{33}$$

The first step in the proof is to use Grothendieck's inequality to prove (33) when $X = \ell_\infty^m$ and $Y = \ell_1^m$ for some m. Having done that, the case where X is a $C(K)$ space and Y is an $L_1(\mu)$ spaces follows immediately by approximation. Finally, we use the lattice theory discussed in section 5 to reduce the general case to the case when $X = C(K)$ and $Y = L_1(\mu)$.

Before proving (33), observe that since $L_1(\mu)$ spaces are 2-concave we can deduce the following from (33): **If T is an operator from a $C(K)$ space into an $L_1(\mu)$ space, then T is 2-summing** and $\pi_2(T) \leq K_G\|T\|$. Since $L_p(\mu)$, $1 \leq p \leq 2$, embeds isometrically into $L_1(\nu)$ for some ν, the local nature of the 2-summing norm gives: **Every operator T from an $\mathcal{L}_{\infty,\lambda}$ space into a subspace of a $\mathcal{L}_{p,\tau}$ space, $1 \leq p \leq 2$, is 2-summing** with $\pi_2(T) \leq \lambda\tau K_G\|T\|$.

We turn to the proof of (33) when T is an operator from ℓ_∞^m to ℓ_1^m. Let $\{e_i\}_{i=1}^m$ be the unit vector basis for ℓ_∞^m and let $x_k = \sum_{i=1}^m b_{ki}e_i$ be in ℓ_∞^m with $\left\|\left(\sum_{k=1}^n |x_k|^2\right)^{1/2}\right\|_\infty \leq 1$. Let $\{u_i\}_{i=1}^n$ be orthonormal in some Hilbert space H, set for $1 \leq i \leq m, y_i = \sum_{k=1}^n b_{ki}u_k$, and define in $\ell_\infty^m(H)$ the vector $\tilde{x} := (y_1, \ldots, y_m)$. Then $\|\tilde{x}\|_{\ell_\infty^m(H)} = \left\|\left(\sum_{k=1}^n |x_k|^2\right)^{1/2}\right\|_\infty \leq 1$. Similarly,

$$\left\|\left(\sum_{k=1}^n |Tx_k|^2\right)^{1/2}\right\|_1 = \|(T \otimes I_H)\tilde{x}\|_{\ell_1^m(H)}$$

and this last quantity is at most $K_G\|T\|$ by (the conceptual form of) Grothendieck's inequality. This gives (33) when $X = \ell_\infty^m$, $Y = \ell_1^m$ and hence also when $X = C(K)$, $Y = L_1(\mu)$.

Suppose now that X and Y are general Banach lattices and $T : X \to Y$ has norm one. Let $x_1, \ldots, x_n$ be in X with $\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\| = 1$ and set $u = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$. Since we are interested only in estimating $\left\|\left(\sum_{i=1}^\infty |Tx_i|^2\right)^{1/2}\right\|$, we can assume by replacing X by the (separable) sublattice generated by $x_1, \ldots, x_n$ that X is separable. The space TX is then separable, so we can similarly assume that Y is separable. Given any $\epsilon > 0$, as seen in section 5 there is a strictly positive functional $y^* \in Y^*$ with

$$y^*\left(\sum_{i=1}^n |Tx_i|^2\right)^{1/2} \geq \left\|\left(\sum_{i=1}^n |Tx_i|^2\right)^{1/2}\right\|$$

and $\|y^*\| \leq 1 + \epsilon$. As mentioned in section 5, the space X_u is lattice isometric to a $C(K)$ space and Y_{y^*} is lattice isometric to an $L_1(\mu)$ space. The natural lattice homomorphisms $J_1 : X_u \to X$, $J_2 : Y \to Y_{y^*}$ satisfy $\|J_1\| = \|u\| = 1$ and $\|J_2\| = \|y^*\| \leq 1 + \epsilon$. We can then apply (33) to the operator J_2TJ_1 to obtain that

$$\left\| \left(\sum_{i=1}^{n} |Tx_i|^2 \right)^{1/2} \right\|_{Y_{y^*}} \leq K_G \|J_2 T J_1\| \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|_{X_u}$$
$$\leq (1+\epsilon) K_G \|T\|.$$

Since

$$\left\| \left(\sum_{i=1}^{n} |Tx_i|^2 \right)^{1/2} \right\|_{Y_{y^*}} = y^* \left(\left(\sum_{i=1}^{n} |Tx_i|^2 \right)^{1/2} \right) \geq \left\| \left(\sum_{i=1}^{n} |Tx_i|^2 \right)^{1/2} \right\|,$$

this completes the proof.

We give now one of the many applications of the preceding results to the geometry of Banach spaces. **Every space X which is isomorphic to a quotient space of an $\mathcal{L}_\infty$ space and is as well a subspace of an $\mathcal{L}_1$ space must be isomorphic to a Hilbert space**. Indeed, if we compose the map from an $\mathcal{L}_\infty$ space into X with the embedding of X into an $\mathcal{L}_1$ space we get an operator from an $\mathcal{L}_\infty$ space to an $\mathcal{L}_1$ space which is thus 2-summing. Like any 2-summing operator this operator factors through a Hilbert space. It follows that X is a quotient space of a Hilbert space and thus isomorphic to a Hilbert space.

A quantitative inspection of the argument above shows that if X is a quotient of a $C(K)$ space and a subspace of an L_1 space then $d(X, \ell_2) \leq K_G$ (provided X is separable and infinite dimensional). Note that ℓ_2 is isometric to a subspace of $L_1(0,1)$ (see section 4) and thus $\ell_2^* = \ell_2$ is also a quotient space of $C(0,1)$ because ℓ_2 is reflexive and $C(0,1)$, considered as a subspace of $L_\infty(0,1) = L_1(0,1)^*$, determines the norm of $L_1(0,1)$. (If Y is a reflexive subspace of a Banach space X and Z is a norm closed subspace of X^* which determines the norm of X, then the restriction operator $z^* \mapsto z^*_{|Y}$ is a quotient mapping from Z onto Y^*.)

We will next give a glimpse into the connections of p-summing operators to the theory of finite dimensional spaces discussed in section 8. If X is the space ℓ_2^n then $\pi_2(I_X) = \sqrt{n}$ since this is the Hilbert-Schmidt norm of the identity in ℓ_2^n. It is quite surprising that: **For any X with** $\dim X = n, \pi_2(I_X) = \sqrt{n}$. To see this take any norm one operator V from ℓ_2 into X. By projecting onto the orthogonal complement of the kernel of V, we can factor V as $V_1 P$ where $V_1 : \ell_2^m \to X$, $P : \ell_2 \to \ell_2^m$, $\|P\| = 1$, $\|V_1\| = \|V\|$, and $m \leq n$. Then

$$\pi_2(I_X V) \leq \|I_X\| \ \|V_1\| \pi_2(I_{\ell_2^m}) \|P\| \leq \sqrt{n}.$$

Taking the supremum over all such V gives $\pi_2(I_X) \leq \sqrt{n}$.

To prove the other inequality, we get from the Pietsch factorization theorem a probability measure μ on B_{X^*} and $T : L_2(\mu) \to X$ so that $T I_{2|_X} = I_X$ and

$\|T\| = \pi_2(I_X)$, where as usual $I_2 : C(B_{X^*}) \to L_2(\mu)$ is the formal identity and X is canonically embedded into $C(B_{X^*})$. The space $X_2 := I_2 X$ is an n-dimensional Hilbert space and $I_2 T$ is the identity on X_2. Hence

$$\sqrt{n} = \pi_2(I_{X_2}) \le \pi_2(I_2)\|T\| = \pi_2(I_X).$$

As a consequence of the preceding we deduce: **The projection constant of an n-dimensional space X is at most $\sqrt{n}$**; that is, whenever Y contains X there is a projection P from Y onto X with $\|P\| \le \sqrt{n}$. Indeed, since $\pi_2(I_X) = \sqrt{n}$, the Pietsch factorization theorem yields that the identity operator on X can be represented as $X \xrightarrow{J} L_\infty(\mu) \xrightarrow{I_2} L_2(\mu) \xrightarrow{V} X$, with J an isometry, I_2 the formal identity, and $\|V\| = \sqrt{n}$. Since $L_\infty(\mu)$ is 1-injective the operator J can be extended to a norm one operator T from Y into $L_\infty(\mu)$. The operator $P := V I_2 T$ is a projection from Y onto X with norm at most $\sqrt{n}$. Notice also that the factorization used above gives another proof of the result proved in section 8 that $d(X, \ell_2^n) \le \sqrt{n}$.

The estimate of $\sqrt{n}$ for the projection constant of an n-dimensional space is essentially sharp. This is discussed in [33].

The *p-integral operators* form a class of operators which are closely related to the p-summing operators. An operator $T : X \to Y$ is said to be p-integral, $1 \le p \le \infty$, (in symbols $T \in \mathcal{I}_p(X,Y)$) provided that the composition $J_Y T$ of T with the canonical embedding $J_Y : Y \to Y^{**}$ factors through the formal identity $I_{\infty,p} : L_\infty(\mu) \to L_p(\mu)$ for some probability measure μ:

$$\begin{array}{ccc} L_\infty(\mu) & \xrightarrow{\quad I_{\infty,p} \quad} & L_p(\mu) \\ {\scriptstyle A}\uparrow & & \downarrow{\scriptstyle B} \\ X & \xrightarrow{T} \; Y \; \xrightarrow{J_Y} & Y^{**} \end{array} \tag{34}$$

The p-integral norm $i_p(T)$ is then defined to be the infimum over all such factorizations of $\|A\|\|B\|$. By taking ultraproducts one sees that this infimum is really a minimum. The space $(\mathcal{I}_p(X,Y), i_p)$ is easily seen to be a Banach space and i_p satisfies the ideal property $i_p(STU) \le \|S\| i_p(T) \|U\|$.

If T is in $\mathcal{I}_p(X,Y)$ and X is a subspace of $C(K)$, with K a compact Hausdorff space, then there is a probability measure ν on K and an operator $S : L_p(\nu) \to Y^{**}$ with $\|S\| = i_p(T)$ which makes the following diagram commute:

$$\begin{array}{ccc} C(K) & \xrightarrow{\quad I_p \quad} & L_p(\nu) \\ \cup & & \downarrow{\scriptstyle S} \\ X & \xrightarrow{T} \; Y \; \xrightarrow{J_Y} & Y^{**} \end{array} \tag{35}$$

Indeed, if (34) holds, A can be extended to an operator $\tilde{A} : C(K) \to L_\infty(\mu)$

because $L_\infty(\mu)$ is 1-injective. By the Pietsch factorization theorem there is a probability measure ν on K so that for each $x \in C(K)$,

$$\|BI_{\infty,p}\tilde{A}x\| \leq \pi_p(BI_{\infty,p}\tilde{A}) \left(\int_K |x|^p \, d\nu \right)^{1/p}$$

and the desired conclusion follows from

$$\pi_p(BI_{\infty,p}\tilde{A}) \leq \|B\|\pi_p(I_{\infty,p})\|\tilde{A}\| = \|B\|\|A\|.$$

It is evident that $\pi_p(T) \leq i_p(T)$ and from the Pietsch factorization theorem it follows that $\pi_p(T) = i_p(T)$ if the domain of T is a $C(K)$ space. As was mentioned implicitly in the discussion of 2-summing operators, $\pi_2(T) = i_2(T)$ for any operator. One reason for defining p-integral via a factorization of $J_Y T$ rather than T is that this forces $i_p(T) = i_p(T^{**})$ (use the fact that a dual space is norm one complemented in its bidual).

The 1-injectivity of $C(K)^{**}$ gives that a p-summing operator T into a $C(K)$ space is p-integral with $i_p(T) = \pi_p(T)$. The equality $i_1(T^*) = i_1(T)$ follows from the observation that the adjoint I_1^* of $I_1 : C(K) \to L_1(\nu)$ is $I_{\infty,1} : L_\infty(\nu) \to L_1(\nu)$ followed by the identification of $L_1(\nu)$ with the norm one complemented subspace of $C(K)^*$ consisting of the finite signed measures which are absolutely continuous with respect to ν.

For other values of p, the adjoint of a p-integral operator need not be strictly singular (see [9, 5.12]) and hence need not be q-summing for any $q < \infty$.

For each $1 \leq p \leq \infty$, $p \neq 2$, there exist p-summing operators which are not p-integral (see [9, 5.13]). The case $p = 1$ is particularly easy to deduce from the theory we have presented. We saw that every operator $T : \ell_1 \to \ell_2$ is 1-summing. If $T : \ell_1 \to \ell_2$ is 1-integral, then T has a factorization $\ell_1 \xrightarrow{A} L_\infty(\mu) \xrightarrow{I_{\infty,1}} L_1(\mu) \xrightarrow{B} \ell_2$. But then B and also $I_{\infty,1}$ are 1-summing, hence $BI_{\infty,1}$, whence also $BI_{\infty,1}A = T$, are compact (use the fact that $L_1(\mu)$ has the DP property).

For $p = \infty$ and Y reflexive the infinity integral operators from X to Y are exactly those which factor through some $L_\infty(\mu)$ space (with the integral norm equal to the best factorization). In particular for X reflexive the infinity integral norm of the identity of X is finite if and only if X is finite dimensional (and is equal to the projection constant of X in that case). Thus also for $p = \infty$ it is evident that summing and integral norms can be very different.

The main reason for introducing p-integral operators is that they are needed for the duality theory of $\Pi_p(X, Y)$. For simplicity, we restrict to the case where X and Y are finite dimensional. Following the notation used in section 8, for finite dimensional X, Y and α a norm on $B(X,Y)$ we represent the dual of $(B(X,Y),\alpha)$ as $(B(Y,X),\alpha^*)$, where the pairing is given by $\langle S,T\rangle = \operatorname{trace} TS \,(= \operatorname{trace} ST)$. Then for all $1 \le p \le \infty$, $\Pi_p(X,Y)^* = \mathcal{I}_{p^*}(Y,X)$ when X and Y are finite dimensional. This just means that for each $S \in B(Y,X)$, $i_{p^*}(S) = \sup\{\operatorname{trace} TS : T \in B(X,Y), \pi_p(T) \le 1\}$.

In view of the discussion in section 8 that $(B(X,Y),\|\cdot\|)^* = \mathcal{N}(Y,X)$ for finite dimensional spaces, we have (since $\pi_\infty = \|\cdot\|$) that the identity $i_1(S) = \mathcal{N}(S)$ for operators between finite dimensional spaces is equivalent to the assertion that $\pi_\infty^* = i_1$. To prove that $i_1(S) = \mathcal{N}(S)$, first note (it is more or less immediate from the definition of $\mathcal{N}(S)$) that $\mathcal{N}(S)$ is the infimum of $\|A\|\|\Delta\|\|B\|$ over all factorizations of S of the form

$$\begin{array}{ccc} \ell_\infty^n & \xrightarrow{\Delta} & \ell_1^n \\ {\scriptstyle A}\uparrow & & \downarrow{\scriptstyle B} \\ Y & \xrightarrow{S} & X \end{array} \tag{36}$$

with Δ a diagonal operator. From this it follows easily that $i_1(S) \le \mathcal{N}(S)$. For the reverse inequality, given a factorization

$$\begin{array}{ccc} L_\infty(\mu) & \xrightarrow{I_{\infty,1}} & L_1(\mu) \\ {\scriptstyle A_1}\uparrow & & \downarrow{\scriptstyle B_1} \\ Y & \xrightarrow{S} & X \end{array} \tag{37}$$

one has for $\epsilon > 0$ an operator $A_\epsilon : Y \to L_\infty(\mu)$ with $\|A_1 - A_\epsilon\| < \epsilon$ so that $A_\epsilon Y$ is contained in the simple functions. This gives a factorization of $B_1 I_{\infty,1} A_\epsilon$ of the form (36) with $\|B\|\|\Delta\|\|A\| \le \|B_1\|\|I_{\infty,1}\|\|A_\epsilon\|$. Setting $N := \dim Y$, we get $\mathcal{N}(S - B_1 I_{\infty,1} A_\epsilon) \le N\|S - B_1 I_{\infty,1} A_\epsilon\| \le N\|A_1 - A_\epsilon\| < N\epsilon$. Letting $\epsilon \to 0$, we conclude that $\mathcal{N}(S) \le i_1(S)$.

Having seen that $i_1(S) = \mathcal{N}(S)$ for operators between finite dimensional spaces, one can easily deduce the identity $\pi_p^* = i_{p^*}$ for $1 < p < \infty$ from the composition inequality $i_1(ST) \le \pi_p(T) i_{p^*}(S)$. In fact, a more general inequality is true for operators between general spaces: **If** $T \in \mathcal{I}_p(X,Y)$ **and** $S \in \Pi_q(Y,Z)$ **then** $ST \in \mathcal{I}_r(X,Z)$ **and** $i_r(ST) \le i_p(T)\pi_q(S)$, **where** $\frac{1}{r} = 1 \wedge (\frac{1}{p} + \frac{1}{q})$. This inequality follows from the composition inequality for p-summing operators mentioned in section 10. Indeed, consider the following commutative diagram:

$$\begin{array}{ccccccc} C(K) & & \xrightarrow{\quad I_p \quad} & & L_p(\mu) & & \\ {\scriptstyle A}\uparrow & & & & \downarrow{\scriptstyle B} & & \\ X & \xrightarrow{T} & Y & \xrightarrow{J_Y} & Y^{**} & \xrightarrow{S^{**}} & Z^{**} \end{array}$$

The operator $S^{**}BI_p$ satisfies

$$i_r(S^{**}BI_p) = \pi_r(S^{**}BI_p) \leq \pi_p(BI_p)\pi_q(S^{**}) \leq \|B\|\pi_p(I_p)\pi_q(S) = \|B\|\pi_q(S)$$

$$\text{and} \quad i_r(J_Z ST) = i_r(S^{**}J_Y T) = i_r(S^{**}BI_pA) \leq \|B\|\pi_q(S)\|A\|.$$

The desired composition inequality follows by taking the infimum over p-integral factorizations of T.

A finite dimensional Banach space X is said to have *enough symmetries* provided that the only operators on X which commute with each isometry on X are the scalar multiples of the identity operator. Finite dimensional spaces with a 1-symmetric basis are examples of spaces with enough symmetries. On the other hand, $B(\ell_2^n, \ell_2^n)$ has enough symmetries but, as noted in section 9, does not even have a "good" unconditional basis.

In general, ideal norms for the operators on a space with enough symmetries are better behaved than on general finite dimensional spaces. For example, if X has enough symmetries, the operator one obtains from applying Lewis' lemma to an ideal norm on $B(X,X)$ is a scalar multiple of I_X. This is a consequence of the following: **If** $\dim X = n$ **and** X **has enough symmetries and** α **is an ideal norm, then** $\alpha(I_X)\alpha^*(I_X) = n$**.** The inequality $\alpha(I_X)\alpha^*(I_X) \geq \operatorname{trace} I_X = n$ is clear. To see the reverse inequality, take $T \in B(X,X)$ so that $\operatorname{trace} T = n$ and $\alpha(I_X)\alpha^*(T) = n$. Let G be the (compact) group of isometries of X and μ normalized Haar measure on G. For each S in G, $\operatorname{trace} S^{-1}TS = \operatorname{trace} T = n$ and, by the ideal property of α^*, $\alpha^*(S^{-1}TS) = \alpha^*(T)$. The operator

$$T_0 := \int_G S^{-1}TS \mathrm{d}\mu(S)$$

satisfies $\operatorname{trace} T_0 = \operatorname{trace} T$, $\alpha^*(T_0) \leq \alpha^*(T)$, and T commutes with all elements of G. Hence $T_0 = \lambda I_X$ and since $\operatorname{trace} T_0 = n$ we conclude that $\lambda = 1$. It follows that $\alpha(I_X)\alpha^*(I_X) \leq n$ and thus we have equality.

A particular case of this last result is worth singling out: **If** $\dim X = n$ **and** X **has enough symmetries then the projection constant of** X **is equal to** $n/\pi_1(I_X)$**.**

11 Interpolation

In this section we give a very sketchy introduction to modern interpolation theory and mention a few applications to Banach space theory. The basic theory is exposed in the books [4] as well as [13] and [5]. The article [31] gives an overview of the subject and further connections to Banach space theory.

We begin by describing the abstract framework of interpolation theory. A *Banach couple* is a pair $\overline{X} = (X_0, X_1)$ of Banach spaces which are contained in a Hausdorff topological vector space Z such that both injections $J_i\colon X_i \to Z$, $i = 0, 1$, are continuous. The ambient space Z can be replaced with the algebraic sum $\Sigma(\overline{X}) := X_0 + X_1 \subset Z$ topologized with the (complete) norm

$$\|x\| = \inf\{\|x_0\|_{[0]} + \|x_1\|_{[1]}\colon\ x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

where $\|\cdot\|_{[i]}$ is the norm of X_i. We always take for Z the space $\Sigma(\overline{X})$. The space $\Delta(\overline{X}) := X_0 \cap X_1$ also plays a role in the theory; it is naturally normed by the (complete) norm

$$\|x\| = \|x\|_{[0]} \vee \|x\|_{[1]}.$$

Thus we have for $i = 0, 1$ inclusions

$$\Delta(\overline{X}) \overset{H_i}{\hookrightarrow} X_i \overset{J_i}{\hookrightarrow} \Sigma(\overline{X})$$

with $\|H_i\| \le 1$, $\|J_i\| \le 1$. Any Banach space X satisfying $\Delta(\overline{X}) \subset X \subset \Sigma(\overline{X})$ with both inclusions continuous is called an *intermediate space* between X_0 and X_1 (or an intermediate space with respect to $\overline{X}$). The spaces X_0, X_1, $\Delta(\overline{X})$, and $\Sigma(\overline{X})$ are all intermediate spaces with respect to $\overline{X}$.

Given Banach couples $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$, and a linear mapping $T\colon\ \Sigma(\overline{X}) \to \sum(\overline{Y})$, we write $T \in B(\overline{X}, \overline{Y})$ provided $T_{|X_i} \in B(X_i, Y_i)$ for $i = 0, 1$.

If X and Y are intermediate spaces with respect to $\overline{X}$ and $\overline{Y}$, respectively, we say that X and Y are *an interpolation pair* for $\overline{X}$ and $\overline{Y}$ provided that if $T \in B(\overline{X}, \overline{Y})$, then $T_{|X} \in B(X, Y)$. If always

$$\|T\|_{X,Y} \le \|T\|_{X_0,Y_0} \vee \|T\|_{X_1,Y_1}$$

X and Y are said to be *an exact interpolation pair* for $\overline{X}$ and $\overline{Y}$, where $\|S\|_{X,Y} := \|S_{|X}\|_{B(X,Y)}$ (if $SX \not\subset Y$ then $\|S\|_{X,Y} := \infty$). Finally, if $0 \le \theta \le 1$ is such that the inequality

$$\|T\|_{X,Y} \le C\|T\|_{X_0,Y_0}^{1-\theta} \cdot \|T\|_{X_1,Y_1}^{\theta}$$

always holds, then the interpolation pair X and Y are said to be *of exponent* θ (and *exact of exponent* θ if $C = 1$). When $\overline{X} = \overline{Y}$ and $X = Y$, we abbreviate by saying e.g. that X is an interpolation space with respect to $\overline{X}$ provided X and X are an interpolation pair with respect to $\overline{X}$ and $\overline{X}$.

The most classical realization of this abstract set-up occurs in the scale of $L_p(\mu)$ spaces. For $1 \le p_0,\ p_1 \le \infty$, the pair $(L_{p_0}(\mu), L_{p_1}(\mu))$ is a Banach couple, and for $p_0 \wedge p_1 \le p \le p_0 \vee p_1$ the space $L_p(\mu)$ is an intermediate space between $L_{p_0}(\mu)$ and $L_{p_1}(\mu)$. In the language of interpolation theory the *Riesz–Thorin interpolation theorem* can be stated as follows. **Let $1 \le p_0, p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$, and $0 < \theta < 1$. Define p and q by**

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then for all measures μ and ν, (complex) $L_p(\mu)$ and $L_q(\nu)$ are an exact interpolation pair of exponent θ with respect to $(L_{p_0}(\mu), L_{p_1}(\mu))$ and $(L_{q_0}(\nu), L_{q_1}(\nu))$. It is formal to derive from the Riesz–Thorin theorem that its statement remains true in the setting of real scalars as long as $p_0 \le p_1$ and $q_0 \le q_1$ (or, what is the same, if $p_1 \le p_0$ and $q_1 \le q_0$). In other cases the word "exact" must be deleted from the statement of the Riesz–Thorin theorem when the scalars are real.

The modern approach to the Riesz–Thorin theorem proceeds via a construction called the *complex method.* Let $\overline{X} = (X_0, X_1)$ be a Banach couple of complex Banach spaces and let $\mathcal{H}(\overline{X})$ denote the space of functions f from the strip $S := \{z \in \mathbb{C}\colon\ 0 \le \text{Re } z \le 1\}$ into $\Sigma(\overline{X})$ which are bounded, continuous, and analytic in the interior S_0 of S, and so that for $j = 0, 1$ the restriction of f to the line $j + i\mathbb{R}$ is a bounded, continuous function into $X_{[j]}$. The space $\mathcal{H}(\overline{X})$ is normed by

$$\|f\|_{\mathcal{H}(X)} = \sup_{t\in\mathbb{R}} \|f(it)\|_{[0]} \vee \|f(1+it)\|_{[1]}.$$

The norm $\|\cdot\|_{\mathcal{H}(\overline{X})}$ is a complete norm on $\mathcal{H}(\overline{X})$ ([4, 4.1.1]). Indeed, by the maximum modulus principle, for $z \in S$

$$\|f(z)\|_{\Sigma(\overline{X})} \le \sup_{t\in\mathbb{R}} \|f(it)\|_{\Sigma(\overline{X})} \vee \|f(1+it)\|_{\Sigma(\overline{X})} \le \|f\|_{\mathcal{H}(\overline{X})}.$$

Using the completeness of $\Sigma(\overline{X})$, it is easy to check that if $\sum_{n=1}^{\infty} \|f_n\|_{\mathcal{H}(\overline{X})} < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges in $\mathcal{H}(\overline{X})$.

For $0 < \theta < 1$ define $\overline{X}_{[\theta]}$ (or simply $X_{[\theta]}$) to be all x in $\Sigma(\overline{X})$ for which $x = f(\theta)$ for some $f \in \mathcal{H}(\overline{X})$ with the norm

$$\|x\|_{[\theta]} := \inf\{\|f\|_{\mathcal{H}(\overline{X})}\colon\ f(\theta) = x\} < \infty.$$

The space $(X_{[\theta]}, \|\cdot\|_{[\theta]})$ is in a natural way isometric to the quotient $\mathcal{H}(\overline{X})/N_\theta(\overline{X})$, where $N_\theta(X) := \{f \in \mathcal{H}(\overline{X})\colon\ f(\theta) = 0\}$ is the kernel of the operator $\mathcal{H}(\overline{X}) \to \sum(\overline{X})$ defined by $f \mapsto f(\theta)$, which as noted above has norm at most one. Consequently $X_{[\theta]} \subset \Sigma(\overline{X})$ with the inclusion of norm ≤ 1. To see that $\Delta(\overline{X}) \subset X_{[\theta]}$ with inclusion of norm ≤ 1, take any x in $\Delta(\overline{X})$ and define $f \in \mathcal{H}(\overline{X})$ by $f(z) \equiv x$. Then $f(\theta) = x$ and for $t \in \mathbb{R}$ we have

$$\|f(it)\|_{[0]} = \|x\|_{[0]}, \quad \|f(1+it)\|_{[1]} = \|x\|_{[1]}$$

so that

$$\|x\|_{[0]} \leq \|f\|_{\mathcal{H}(\overline{X})} \leq \|x\|_{[0]} \vee \|x\|_{[1]} = \|x\|_{\Delta(\overline{X})}.$$

Thus $X_{[\theta]}$ is an intermediate space with respect to $\overline{X}$. Moreover, **if $\overline{X}$ and $\overline{Y}$ are Banach couples and $0 < \theta < 1$, then $X_{[\theta]}$ and $Y_{[\theta]}$ are an exact interpolation pair of exponent θ with respect to $\overline{X}$ and $\overline{Y}$.** To see this, let $T \in B(\overline{X}, \overline{Y})$, $x \in X_{[\theta]}$, $\varepsilon > 0$, and choose $f \in \mathcal{H}(\overline{X})$ so that $f(\theta) = x$ with $\|f\|_{\mathcal{H}(\overline{X})} \leq \|x\|_{[\theta]} + \varepsilon$. Define $g\colon\ S \to \sum(\overline{Y})$ by

$$g(z) = \|T\|_{X_0,Y_0}^{z-1} \|T\|_{X_1,Y_1}^{-z} Tf(z).$$

Then g is in $\mathcal{H}(\overline{Y})$, $\|g\|_{\mathcal{H}(\overline{Y})} \leq \|f\|_{\mathcal{H}(\overline{X})}$, and $g(\theta) = \|T\|_{X_0,Y_0}^{\theta-1} \|T\|_{X_1,Y_1}^{-\theta} Tx$ so that

$$\begin{aligned} \|Tx\|_{[\theta]} \leq & \quad \|T\|_{X_0,Y_0}^{1-\theta} \|T\|_{X_1,Y_1}^{\theta} \|g\|_{\mathcal{H}(\overline{Y})} \\ & \leq \|T\|_{X_0,Y_0}^{1-\theta} \|T\|_{X_1,Y_1}^{\theta} (\|x\|_{[\theta]} + \varepsilon). \end{aligned}$$

In order to derive the Riesz–Thorin theorem from this last result, it suffices to check that $(L_{p_0}(\mu), L_{p_1}(\mu))_{[\theta]}$ can be identified with $L_p(\mu)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. First suppose that $x \in L_p(\mu)$ with $\|x\|_p = 1$. Define $f \in \mathcal{H} := \mathcal{H}(L_{p_0}(\mu), L_{p_1}(\mu))$ by

$$f(z) = |x|^{p/p(z)} \frac{x}{|x|}$$

where $\frac{1}{p(z)} := \frac{(1-z)}{p_0} + \frac{z}{p_1}$ and $\frac{0}{0} := 0$. Evidently $\|f\|_{\mathcal{H}} \leq 1$ and $f(\theta) = x$, so $x \in (L_{p_0}(\mu), L_{p_1}(\mu))_{[\theta]}$ and $\|x\|_{[\theta]} \leq 1 = \|x\|_p$.

Next suppose that $x \in (L_{p_0}(\mu), L_{p_1}(\mu))_{[\theta]}$, $\|x\|_{[\theta]} < 1$, and $f \in \mathcal{H}$ satisfies $\|f\|_{\mathcal{H}} < 1$, $f(\theta) = x$. To see that $\|x\|_p \leq 1$ it is sufficient to check that if y is a μ-measurable function with $\mu[y \neq 0] < \infty$ and $\|y\|_{p^*} \leq 1$, then $|\langle x, y\rangle| := \int |xy| d\mu \leq 1$. For $z \in S$ set

$$g(z) = |y|^{p^*/p^*(z)} \frac{y}{|y|},$$

where $\frac{1}{p^*(z)} := \frac{1-z}{p_0^*} + \frac{z}{p_1^*}$, and define $h(z) = \langle f(z), g(z)\rangle$. Then since $\|f\|_{\mathcal{H}} < 1$, for $t \in \mathbb{R}$ we have $|h(it)| \vee |h(1+it)| \leq 1$. By the maximum modulus principle we conclude that $|\langle x, y\rangle| = |h(\theta)| \leq 1$ and hence $|\langle x, y\rangle| \leq 1$.

Of the many classical applications of the Riesz–Thorin theorem we mention only one, the *Hausdorff–Young theorem*: **Let $\mathcal{F}$ be the Fourier transform for functions on $\mathbb{R}^n$, defined by $\mathcal{F}f(s) = \int_{\mathbb{R}^n} f(t) e^{-i\langle t,s\rangle} \mathrm{d}t$. Then $\mathcal{F}$ maps $L_p(\mathbb{R}^n)$ into $L_{p^*}(\mathbb{R}^n)$ for $1 \leq p \leq 2$.** To prove the Hausdorff–Young theorem, just observe that $\|\mathcal{F}\|_{L_1, L_\infty} = 1$ and $\|\mathcal{F}\|_{L_2, L_2} = 2\pi^{\frac{n}{2}}$, so that for $1 \leq p \leq 2$, $\|\mathcal{F}\|_{L_p, L_{p^*}} \leq 2^{\frac{2}{p^*}} \pi^{\frac{n}{p^*}}$ by the Riesz–Thorin theorem.

There are also useful vector-valued versions of the Riesz–Thorin theorem. These can be deduced from the following. **Let (X_0, X_1) be a Banach couple, $1 \leq p_0, p_1 < \infty$, $0 < \theta < 1$, and set $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then**

$$(L_{p_0}(\mu, X_0), L_{p_1}(\mu, X_1))_{[\theta]} = L_p(\mu, (X_0, X_1)_{[\theta]}).$$

If $p_0 < \infty$, then also

$$(L_{p_0}(\mu, X_0), L^0_\infty(\mu, X_1))_{[\theta]} = L_p(\mu, X_0, X_1)_{[\theta]},$$

where $L^0_\infty(\mu, X)$ is the closure in $L_\infty(\mu, X)$ of the simple functions. The proof of this theorem (see [4, 5.1.2]) is only a bit more complicated than the argument given above that $(L_{p_0}(\mu), L_{p_1}(\mu))_{[\theta]} = L_p(\mu)$. The argument does use the following general fact about the complex method. **For any Banach couple $\overline{X}$ and $0 < \theta < 1$, $\Delta(\overline{X})$ is dense in $\overline{X}_{[\theta]}$** (see [4, 4.2.2]).

The identification $L_p = (L_{p_0}, L_{p_1})_{[\theta]}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, has a generalization to Banach lattices. Assume that X_0 and X_1 are complex Banach lattices of μ-measurable functions which are ideals in the space of all μ-measurable functions, $0 < \theta < 1$, and set

$$X_0^{1-\theta} X_1^\theta := \{(\text{sign } x_0 x_1)|x_0|^{1-\theta}|x_1|^\theta : x_0 \in X_0,\, x_1 \in X_1\}.$$

If X_0 is order continuous then $(X_0, X_1)_{[\theta]} = X_0^{1-\theta} X_1^\theta$ and

$$\|x\|_{[\theta]} = \inf\{\|x_0\|_{[0]}^{1-\theta} \|x_1\|_{[1]}^\theta\colon\ |x| = |x_0|^{1-\theta}|x_1|^\theta\}. \tag{38}$$

For a proof see section iv.1.11 in [13]. The inequality $\leq$ in (38); namely,

$$\||x_0|^{1-\theta}|x_1|^{\theta}\|_{[\theta]} \leq \|x_0\|_{[0]}^{1-\theta}\|x_1\|_{[1]}^{\theta},$$

does not require order continuity of X_0. It reduces to the lattice Hölder inequality (4) when $X_0 = X_1$.

Since $X_0^{1-\theta}L_\infty(\mu)^\theta = X_0^{1-\theta} = X_0^{(\frac{1}{1-\theta})}$ (the $\frac{1}{1-\theta}$-convexification of X_0), it follows that the p-convexification, $1 < p < \infty$, of an order continuous Banach lattice X_0 can be obtained by using the complex method to interpolate X_0 with an appropriate abstract M-space (here there is no problem in passing to the complexification of a real Banach lattice and returning at the end to the real setting).

We turn now to a discussion of another interpolation method, which is a special case of what is usually called the $\mathcal{K}$-*method*. Given a Banach couple $\overline{X} = (X_0, X_1)$ and $0 < a, b < \infty$, let $\mathcal{K}(\cdot, a, b)$ be the norm of the algebraic sum of $(X_0, a\|\cdot\|_{[0]})$ and $(X_1, b\|\cdot\|_{[1]})$; that is,

$$\mathcal{K}(x, a, b) = \inf\{a\|x_0\|_{[0]} + b\|x_1\|_{[1]} \colon\ x = x_0 + x_1, x_0 \in X_0; x_1 \in X_1\}. \quad (39)$$

Here we follow the notation of [15, 2.g] because there applications of the $\mathcal{K}$-method to Banach space theory are described. Usually in interpolation theory one uses only $\mathcal{K}(\cdot, t) := \mathcal{K}(\cdot, 1, t)$; in this notation $\mathcal{K}(\cdot, a, b)$ is just $a\mathcal{K}(\cdot, \frac{b}{a})$.

Let $\overline{X} = (X_0, X_1)$ be a Banach couple. We mention a method for using the $\mathcal{K}$-functional to build interpolation spaces with respect to $\overline{X}$. Let E be a space with a normalized 1-unconditional basis $\{e_n\}_{n=1}^\infty$ and let $\mathbf{a} = \{a_n\}_{n=1}^\infty$, $\mathbf{b} = \{b_n\}_{n=1}^\infty$, be sequences of positive scalars such that $\sum_{n=1}^{\infty} a_n \wedge b_n < \infty$. The space $\mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$ is defined to be all $x \in \Sigma(\overline{X})$ for which $\sum_{n=1}^{\infty} \mathcal{K}(x, a_n, b_n)e_n$ converges in E, normed by

$$\|x\|_{\mathcal{K}(\overline{X}, \mathbf{a}, \mathbf{b})} = \left\| \sum_{n=1}^{\infty} \mathcal{K}(x, a_n, b_n)e_n \right\|_E . \quad (40)$$

It is simple to check that if $\overline{Y}$ is another Banach couple, then $\mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$ and $\mathcal{K}(\overline{Y}, E, \mathbf{a}, \mathbf{b})$ are an exact interpolation pair with respect to $\overline{X}$ and $\overline{Y}$.

Earlier we saw how the complex method can be used to construct the p-convexification of a Banach lattice. The $\mathcal{K}$-method can also be used to construct Banach spaces with interesting properties. Suppose, for example, that both X_0 and X_1 have a normalized 1-unconditional basis, each of which we identify with the unit vectors $\{e_n\}_{n=1}^\infty$ so that X_0 and X_1 are both contained in c_0 and thus $\overline{X} = (X_0, X_1)$ is a Banach couple. Let E be another space with a normalized 1-unconditional basis, also denoted by $\{e_n\}_{n=1}^\infty$. Since $\mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$ is an exact interpolation space with respect to $\overline{X}$, it follows that $\{e_n\}_{n=1}^\infty$ is a 1-unconditional basis for $\mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$ which moreover is even 1-symmetric if $\{e_n\}_{n=1}^\infty$ is 1-symmetric in both X_0 and in X_1 (one need only check that the span of $\{e_n\}_{n=1}^\infty$ is dense in $\mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$). Assume now that $\{e_n\}_{n=1}^\infty$ is 1-symmetric in both X_0 and X_1, $X_0 \subset X_1$, and $\|\sum_{k=1}^n e_k\|_{[0]}^{-1} \|\sum_{k=1}^n e_k\|_{[1]} \to 0$ as $n \to \infty$. It turns out (see [14, 3.b.4]) that if $a_n = b_n^{-1}$ and the weight sequence $\mathbf{b}$ satisfies $b_n \uparrow \infty$ sufficiently quickly, then there are disjoint finite subsets $\mathcal{C}_n$ of $\mathbb{N}$ so that the mapping $e_n \mapsto \|1_{\mathcal{C}_n}\|_{\mathcal{K}(\overline{X}, \mathbf{a}, \mathbf{b})} 1_{\mathcal{C}_n}$ extends to an isomorphism from E onto a complemented subspace of $\mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$. This is how one proves the result mentioned in section 3 that a space with an unconditional basis is isomorphic to a complemented subspace of a space which has a symmetric basis.

"Good" properties (such as reflexivity and superreflexivity) possessed by the Banach spaces X_0 and X_1 (often possessed by just one of the spaces) generally pass to interpolation spaces between X_0 and X_1 which are obtained by the complex method or by the $\mathcal{K}$-method (at least when the weight sequences $\mathbf{a}, \mathbf{b}$ satisfy some growth conditions and the space E is "nice"). Sometimes interpolation spaces even have a good property which neither X_0 nor X_1 possesses. For example, suppose that $X_0 \subset X_1$ (this can be relaxed but is good enough for applications). **If E is reflexive, the inclusion $J\colon\ X_0 \to X_1$ is weakly compact, $\sum_{n=1}^\infty a_n < \infty$, and $b_n \uparrow \infty$, then $\mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$ is reflexive.** For a proof when $E = \ell_2$, which is easily modified to cover the general case, see [15, 2.g.11]. A consequence of this result is the following factorization theorem for weakly compact operators. **If $T : X \to X_1$, is weakly compact then T factors through a reflexive space;** that is, there exists a reflexive Banach space Y and operators $A \in B(X, Y)$, $B \in B(Y, X_1)$ so that $T = BA$. To derive this factorization theorem from the interpolation result mentioned above, it suffices to take $Y = \mathcal{K}(X_0, X_1, \ell_2, \{2^{-n}\}_{n=1}^\infty, \{2^n\}_{n=1}^\infty)$ where X_0 is the span in X of $\overline{TB_X}$ with $\overline{TB_X}$ as the unit ball of X_0. Let A be T, considered as an operator into Y, and let B be the formal inclusion from Y into X_1. Then A and B are operators and $T = BA$.

We turn now to the realization of the $\mathcal{K}$-method which is used most often in analysis and is discussed extensively in the books on interpolation theory we have mentioned. Given a Banach couple $\overline{X} = (X_0, X_1)$, $0 < \theta < 1$, and $1 \le p < \infty$, $\overline{X}_{\theta,p}$ denotes the space $\mathcal{K}(\overline{X}, \ell_p, \mathbf{a}, \mathbf{b})$ where $a_{2n} := e^{\theta n}$, $b_{2n} := e^{-(1-\theta)n}$, $a_{2n+1} := e^{-\theta n}$, $b_{2n+1} := e^{(1-\theta)n}$. Instead of using $\|\cdot\|_{\mathcal{K}(\overline{X}, \mathbf{a}, \mathbf{b})}$ on $X_{\theta,p}$, it is customary to use

$$\|x\|_{\theta,p} = \inf \left(\int_{\mathbb{R}} \|e^{\theta t} x_0(t)\|_{[0]}^p dt \right)^{1/p} \vee \left(\int_{\mathbb{R}} \|e^{-(1-\theta)t} x_1(t)\|_{[1]}^p dt \right)^{1/p} \tag{41}$$

where the infimum is over all $x_0(t)$, $x_1(t)$ for which $e^{\theta t}x_0(t) \in L_p(\mathbb{R}, X_0)$, $e^{-(1-\theta)t}x_1(t) \in L_p(\mathbb{R}, X_1)$, and $x = x_0(t)+x_1(t)$ for every t in $\mathbb{R}$. The expression $\|\cdot\|_{\theta,p}$ is a norm on $\overline{X}_{\theta,p}$ which is equivalent to $\|\cdot\|_{\mathcal{K}(\overline{X},\mathbf{a},\mathbf{b})}$ but is better behaved. For example, $(\overline{X}_{\theta,p}, \|\cdot\|_{\theta,p})$ is uniformly convex (rather than just superreflexive) if either X_0 or X_1 is uniformly convex and $1 < p < \infty$ ([15, 2.g.21]). Moreover, $\|\cdot\|_{\theta,p}$ is very good for interpolation purposes, for if $\overline{Y}$ is another Banach couple then $(\overline{X}_{\theta,p}, \|\cdot\|_{\theta,p})$ and $(\overline{Y}_{\theta,p}, \|\cdot\|_{\theta,p})$ are an exact interpolation pair of exponent θ with respect to $\overline{X}$ and $\overline{Y}$.

It is of course important to identify $\overline{X}_{\theta,p}$ when $\overline{X}$ is a concrete Banach couple. The spaces that arise in this connection when $\overline{X}$ is a couple of $L_p(\mu)$ spaces are the $L_{p,q}$ spaces. Given $0 < p < \infty$, $0 < q < \infty$, a measure μ, and a Banach space X, $L_{p,q}(\mu, X)$ is the space of X valued strongly measurable functions x for which

$$\|x\|_{p,q} := \left(q/p \int_0^\infty |t^{1/p} x^*(t)|^q \frac{dt}{t} \right)^{1/q} < \infty, \tag{42}$$

where $x^*(t)$ is the decreasing rearrangement of $\|x(t)\|_X$. For $0 < p \le \infty$, $L_{p,\infty}(\mu, X)$ is the space of X valued strongly measurable functions for which

$$\|x\|_{p,\infty} := \sup_{t>0} t^{1/p} x^*(t) < \infty. \tag{43}$$

When X is the scalar field we write $L_{p,q}(\mu)$. Evidently $L_{p,p}(\mu, X) = L_p(\mu, X)$. Note that for $p > q \ge 1$, the space $L_{p,q}(\mu)$ is the Lorentz function space $L_{W,q}(\mu)$ defined in section 5 with $W(t) = qt^{q/p-1}$ and hence is a Banach space. When $q > p \ge 1$ the expression $\|\cdot\|_{p,q}$ does not satisfy the triangle inequality, although $\|\cdot\|_{p,q}$ is equivalent to a norm when $p > 1$. In any case $L_{p,q}(\mu, X)$ is a metrizable topological vector space.

The main result about spaces obtained from L_p and $L_{p,q}$ spaces via the $\mathcal{K}$-method is the following (see [4, 5.3.1]): **Let $0 < p_0, p_1, q_0, q_1 \le \infty$, $0 < \theta < 1$, and define p by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If $p_0 \ne p_1$, or if $p_0 = p_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then**

$$(L_{p_0,q_0}(\mu, X), L_{p_1,q_1}(\mu, X))_{\theta,q} = L_{p,q}(\mu, X) \tag{44}$$

up to equivalence of the "norms"; i.e., for some constant $C > 0$, $C^{-1}\|\cdot\|_{p,q} \leq \|\cdot\|_{\theta,q} \leq C\|\cdot\|_{p,q}$. When the expressions $\|\cdot\|_{p_i,q_i}$, $i = 0,1$, are equivalent to norms one can deduce from (44) and earlier comments an interpolation theorem for operators. In fact, there are cases where interpolation is valid even when the spaces are not all Banach spaces. In particular, assume that $T\colon\ L_{p_i,r_i}(\mu, X) \to L_{q_i,s_i}(\nu, X)$ is continuous for $i = 0,1$, with $p_0 \neq p_1$, $q_0 \neq q_1$, $0 < \theta < 1$, and define p, q by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} = \frac{\theta}{q_1}$. If $p \leq q$, then $T\colon\ L_p(\mu, X) \to L_q(\nu, Y)$ is continuous and for $0 \leq r \leq s \leq \infty$, $T\colon\ L_{p,r}(\mu, X) \to L_{q,s}(\nu, Y)$ is continuous (see [4, 5.3.1]).

In some cases it is possible to characterize the interpolation spaces with respect to a concrete Banach couple. The Banach couple $\overline{X} = (X_0, X_1)$ is called a *Calderón couple* if whenever x, y in $\Sigma(\overline{X})$ satisfy $\mathcal{K}(y,1,t) \leq \mathcal{K}(x,1,t)$ for all t, then there is T in $B(\overline{X}, \overline{X})$ so that $Tx = y$. If always T can be chosen so that $\|T\|_{X_0,X_0} \vee \|T\|_{X_1,X_1} \leq 1$, then $\overline{X}$ is called an *exact Calderón couple.* It is easy to see that if X is an intermediate space with respect to a Calderón couple $\overline{X}$, then X is an interpolation space with respect to $\overline{X}$ if and only if it is $\mathcal{K}$*-monotone* for $\overline{X}$, which means that if $x \in X$, $y \in \Delta(\overline{X})$, and $\mathcal{K}(y,1,t) \leq \mathcal{K}(x,1,t)$ for all t, then $y \in X$. It then follows from the $\mathcal{K}$*-divisibility theorem* that X is obtainable from $\overline{X}$ via the $\mathcal{K}$-method. This theorem says the following. **If X is $\mathcal{K}$-monotone for $\overline{X}$, then there are E, a, and b so that $X = \mathcal{K}(\overline{X}, E, \mathbf{a}, \mathbf{b})$, up to an equivalent renorming.**

The pair $(L_p(0,1), L_q(0,1))$, $1 \leq p < q \leq \infty$, is a Calderón couple, and many other examples are known (see [31]). One interesting problem which is not completely solved is to determine a necessary and sufficient condition for a Banach couple of symmetric lattice ideals on $[0,1]$ (see section 5) to be a Calderón couple. Much is known about the interpolation spaces for such a couple. For example, an exact interpolation space with respect to such a couple must itself be a symmetric lattice ideal (the first step is to observe that if $\tau : [0,1] \to [0,1]$ is a measure preserving automorphism, then $x \mapsto x(\tau)$ defines an isometric automorphism of any symmetric lattice on $[0,1]$). When the couple is $(L_1(0,1), L_\infty(0,1))$, it is reasonable to guess that the converse is true. It is not, but: **If $L_\infty(0,1)$ is dense in the symmetric lattice ideal X, then X is an exact interpolation space with respect to $(L_1(0,1), L_\infty(0,1))$** ([13, Th. 4.10]). Most natural symmetric lattice ideals on $[0,1]$, including all the separable ones, satisfy the hypothesis of this theorem.

12 List of symbols

Here is a list of symbols used in this introductory article and, where appropriate, a reference to where they are defined.

$\mathbb{N}$ The natural numbers.

$\mathbb{R}$ The real numbers.

$\mathbb{C}$ The complex numbers.

$\tilde{S}$ The complement of the set S.

$\mathbb{T}$ The unit circle in the complex plane.

$\mathbb{P}$ A probability measure (section 2).

$\mathbb{E}$ $\int \cdot \, \mathrm{d}\mathbb{P}$ (section 2).

A_P, B_p The constants in Khintchine's inequality (1); also the constants in the Kahane-Khintchine inequality (24).

K_G The constant in Grothendieck's inequality (section 10).

$C(K;X)$ Continuous functions f on the (usually) compact Hausdorff space K taking values in the (usually) normed space X, normed by $\|f\| = \sup_{t\in K} \|f(t)\|$.

$C(K)$ $C(K;X)$ when X is the scalar field.

$L_p(\mu, X)$ The μ-measurable X-valued functions f for which $\|f\|_p := (\int \|f\|^p \, \mathrm{d}\mu)^{1/p} < \infty$ (section 7). Here $0 < p < \infty$.

$L_p(\mu)$ $L_p(\mu, X)$ when $X = \mathbb{R}$.

$L_\infty(\mu, X)$ The μ-measurable essentially bounded X-valued functions, with norm $\|f\|_\infty := \inf_{\mu A=0} \sup |f_{|\tilde{A}}|$.

$L_p(0,1)$ $L_p(\mu)$ when μ is Lebesgue measure on the unit interval.

$L_p(\mathbb{T})$ $L_p(\mu)$ when μ is normalized Lebesgue measure on the unit circle.

$\ell_p(\Gamma)$ $L_p(\mu)$ when μ is counting measure on the set Γ.

ℓ_p $\ell_p(\Gamma)$ when $\Gamma = \mathbb{N}$.

ℓ_p^n $\ell_p(\Gamma)$ when $\Gamma = \{1, 2, \ldots, n\}$.

c The subspace of ℓ_∞ of scalar sequences which have a limit.

$c_0(\Gamma)$ The closure in $\ell_\infty(\Gamma)$ of the scalar sequences which have finite support.

c_0 $c_0(\Gamma)$ when $\Gamma = \mathbb{N}$.

$L_{p,q}(\mu, X)$ The μ-measurable X-valued functions f for which $\|f\|_{p,q} := \left(\frac{q}{p}\int_0^\infty \|t^{1/p} f^*(t)\|_X^q \frac{\mathrm{d}}{t}\right)^{1/q} < \infty$ (section 11). Here $0 < p, q < \infty$ and f^* is the decreasing rearrangement of $\|f\|_X$.

$L_{p,\infty}(\mu, X)$ The μ-measurable X-valued functions f for which $\|f\|_{p,\infty} := \sup_{t>0} t^{1/p} \|f^*(t)\|_X < \infty$ (section 11). Here $0 < p \leq \infty$ and f^* is the decreasing rearrangement of $\|f\|_X$.

$L_{p,q}(\mu)$ $L_{p,q}(\mu, X)$ when $X = \mathbb{R}$.

B_X The closed unit ball of the Banach space X.

$B_X(x, r)$ The closed ball of radius r with center x in the Banach space X; denoted also $B(x, r)$ when X is understood.

$S^\perp$ All linear functionals which vanish on S [when S is a subset of a Banach space].

$S_\perp$ The intersection of the kernels of all linear functionals in S [when S is a subset of the dual of a Banach space].

$d(X, Y)$ The Banach-Mazur distance from X to Y (section 2).

$X \approx Y$ The space X is isomorphic to the space Y.

$\delta_X(\cdot)$ The modulus of convexity of the space X (section 6).

$\rho_X(\cdot)$ The modulus of smoothness of the space X (section 6).

X_u When X is a lattice and $u \geq 0$, the abstract M-space which has the order interval $[-u, u]$ as the unit ball (section 5).

X_{u^*} When X is a lattice and $u^* \geq 0$ in X^*, the abstract L_1-space which is the completion of X under the seminorm $\|x\|_{u^*} = u^*(|x|)$ (section 5).

$\{\varepsilon_n\}_{n=1}^\infty$ A Rademacher sequence (section 4).

I_X The identity operator on the space X.

J_X The canonical embedding of X into X^{**}.

I_p The formal identity operator from $C(K)$ to $L_p(\mu)$ (when μ is a finite measure on the compact Hausdorff space K).

$I_{p,q}$ The formal identity mapping from $L_p(\mu)$ to $L_q(\mu)$.

$B(X,Y)$ The bounded operators from X to Y.

$K(X,Y)$ The compact operators from X to Y.

$WK(X,Y)$ The weakly compact operators from X to Y.

$SS(X,Y)$ The strictly singular operators from X to Y (section 10).

$Fr(X,Y)$ The Fredholm operators from X to Y (section 10).

$\mathcal{N}(X,Y)$ The nuclear operators from X to Y (section 8).

$\mathcal{N}(T)$ The nuclear norm of the operator T (section 8).

$M^{(p)}(T)$ The p-convexity constant of the operator T (section 5).

$M_{(p)}(T)$ The p-concavity constant of the operator T (section 5).

$T_p(X)$ The type p constant of the Banach space X (section 8).

$C_p(X)$ The cotype p constant of the Banach space X (section 8).

$\Pi_p(X,Y)$ The p-summing operators from X to Y (section 10).

$\pi_p(T)$ The p-summing norm of the operator T (section 10).

$\mathcal{I}_p(X,Y)$ The p-integral operators from X to Y (section 10).

$i_p(T)$ The p-integral norm of the operator T (section 10).

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